Nonlinear Partial Differential Equations I University of Bonn
Winter term 2016/2017 Prof. Dr. M. Disertori
Problem Sheet 10 (due Monday 16.01.2017) Dr. M. Bonacini

Problem 1 (An application of Schauder's fixed point theorem, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions:

a) for every $s \in \mathbb{R}$ the map $x \mapsto f(x,s)$ is measurable in Ω;

b) for almost every $x \in \Omega$ the map $s \mapsto f(x,s)$ is continuous on $\mathbb{R}$;

c) $|f(x,s)| \leq a(x) + b|s|^\beta$ for every $s \in \mathbb{R}$ and almost every $x \in \Omega$, where $a \in L^2(\Omega)$, $b > 0$, and $0 \leq \beta < 1$.

Use Schauder's fixed point theorem (in its second version) to show that there exists a weak solution $u \in H_0^1(\Omega)$ to the boundary value problem

$$\begin{cases} -\Delta u = f(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If $\beta = 1$, is there always a weak solution to the problem?
Hint: recall the property of Nemitski operators seen in Problem 3 in Problem Sheet 9. Define a map T which associates to every $u \in H_0^1(\Omega)$ the unique solution $v \in H_0^1(\Omega)$ to the equation $-\Delta v = f(x, u(x))\dots$

Problem 2 (Another application of Schauder's fixed point theorem, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $f \in L^2(\Omega)$, and let $a : \mathbb{R} \to \mathbb{R}$ be continuous and such that $\alpha_1 \leq a(s) \leq \alpha_2$ for every $s \in \mathbb{R}$, where $0 < \alpha_1 < \alpha_2 < \infty$. Use Schauder's fixed point theorem (in its second version) to show that there exists a weak solution $u \in H_0^1(\Omega)$ to the boundary value problem

$$\begin{cases} -\mathrm{div}\big(a(u)\nabla u\big) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

that is,

$$\int_\Omega a(u)\nabla u \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x \qquad \text{for every } \varphi \in H_0^1(\Omega).$$

Please turn over.

Problem 3 (Schaefer's fixed point theorem, 4 points).
Let X be a Banach space, and let $T : X \to X$ be an operator such that:

a) T is continuous;

b) T is compact;

c) there exists $R > 0$ such that $\{x \in X : x = \lambda T(x) \text{ for some } \lambda \in [0, 1]\} \subset B_R$,

where B_R is the ball with radius R in X. Prove that there exists $x \in B_R$ such that $x = T(x)$.
Hint: define the projection operator $p_R : X \to \overline{B}_R$ *by*

$$p_R(x) = \begin{cases} x & \textit{if } x \in \overline{B}_R, \\ \frac{x}{|x|} R & \textit{if } x \notin \overline{B}_R, \end{cases}$$

and apply Schauder's fixed point theorem to the map $T_R := p_R \circ T$.

Problem 4 (An application of Schaefer's fixed point theorem, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfy the following assumptions:

a) for every $\xi \in \mathbb{R}^n$ the map $x \mapsto f(x, \xi)$ is measurable in Ω;

b) for almost every $x \in \Omega$ the map $\xi \mapsto f(x, \xi)$ is continuous on $\mathbb{R}^n$;

c) $|f(x, \xi)| \leq a(x) + b|\xi|$, for some $a \in L^2(\Omega)$, $b > 0$.

Use Schaefer's fixed point theorem to show that there exists a weak solution $u \in H_0^1(\Omega)$ to the boundary value problem

$$\begin{cases} -\Delta u + \mu u = f(x, \nabla u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

provided that μ is sufficiently large.
Hint: to prove the second assumption of Schaefer's theorem (compactness), use the fact that the embedding $L^2(\Omega) \hookrightarrow (H_0^1(\Omega))'$ *is compact, since it is the adjoint of the compact embedding* $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ *(Schauder's Theorem). In the proof of the third assumption of Schaefer's Theorem, use Young's inequality* $ab \leq \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}$.

Total: 16 points