Nonlinear Partial Differential Equations I University of Bonn
Winter term 2016/2017 Prof. Dr. M. Disertori
Problem Sheet 9 (due Monday 09.01.2017) Dr. M. Bonacini

Problem 1 (Sub- and supersolution method, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set, and consider the elliptic operator $Lu := -\sum_{i,j=1}^n (a_{ij} u_{x_i})_{x_j}$, where the coefficients $a_{ij} \in L^\infty(\Omega)$ are uniformly elliptic. Let $f \in L^2(\Omega)$ and let

$$g : \mathbb{R} \to \mathbb{R} \quad \text{non decreasing, continuous, with} \quad |g(s)| \le \gamma |s|^\alpha ,$$

for some constants $\gamma > 0$ and $0 \le \alpha \le \frac{n+2}{n-2}$. We want to study the nonlinear boundary value problem

$$\begin{cases} Lu = g(u) + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1}$$

Recall that $\underline{u} \in H_0^1(\Omega)$ is a *weak subsolution* for problem (1) provided

$$\sum_{i,j=1}^n \int_\Omega a_{ij} \underline{u}_{x_i} \varphi_{x_j} \,\mathrm{d}x \le \int_\Omega g(\underline{u}) \varphi \,\mathrm{d}x + \int_\Omega f \varphi \,\mathrm{d}x \qquad \text{for every } \varphi \in H_0^1(\Omega), \varphi \ge 0\,.$$

Similarly, $\bar{u} \in H_0^1(\Omega)$ is a *weak supersolution* for problem (1) provided

$$\sum_{i,j=1}^n \int_\Omega a_{ij} \bar{u}_{x_i} \varphi_{x_j} \,\mathrm{d}x \ge \int_\Omega g(\bar{u}) \varphi \,\mathrm{d}x + \int_\Omega f \varphi \,\mathrm{d}x \qquad \text{for every } \varphi \in H_0^1(\Omega), \varphi \ge 0\,.$$

Assume that there exist a weak subsolution $\underline{u}$ and a weak supersolution $\bar{u}$ such that $\underline{u} \le \bar{u}$ almost everywhere in Ω. Prove that there exists a weak solution $u \in H_0^1(\Omega)$ to (1), satisfying $\underline{u} \le u \le \bar{u}$.
Hint: define inductively $u_0 = \underline{u}$ and $u_k \in H_0^1(\Omega)$, $k = 1, 2, \ldots$, as the unique weak solution to the linear problem

$$\begin{cases} Lu_k = g(u_{k-1}) + f & \textit{in } \Omega, \\ u_k = 0 & \textit{on } \partial\Omega\,. \end{cases}$$

By using the maximum principle for weak solutions (see Problem 1 in Problem Sheet 6) show that $\underline{u} \le u_0 \le u_1 \le \ldots \le u_k \le \ldots \le \bar{u}$. Then show that the sequence $(u_k)_k$ converges to a weak solution to (1).

Problem 2 (Sub- and supersolution method continued, 4 points).
Assume that Ω, L, f are as in Problem 1.

a) Generalized the result in Problem 1 to the elliptic boundary value problem

$$\begin{cases} Lu = h(u) + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where now $h : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, but not necessarily monotone (highlight only the main differences with respect to the proof of Problem 1).

b) Use this method to prove the existence of a weak solution $u \in H_0^1(\Omega)$ to

$$\begin{cases} -\Delta u = h(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

where $h : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, bounded, with $h(0) = 0$, and differentiable at the origin with $h'(0) > \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ in Ω (assume also that $\partial\Omega$ is regular).

Problem 3 (Nemitski operator, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a *Carathéodory function*, that is, f satisfies the following conditions:

a) for every $s \in \mathbb{R}$ the map $x \mapsto f(x, s)$ is measurable in Ω;

b) for almost every $x \in \Omega$ the map $s \mapsto f(x, s)$ is continuous on $\mathbb{R}$.

Let $1 \leq p, q < \infty$, and define the Nemitski composition operator $F : L^p(\Omega) \to L^q(\Omega)$ by setting $F(u)(x) := f(x, u(x))$. Prove that F is well defined and continuous, provided f satisfies

$$|f(x, s)| \leq a(x) + b|s|^{\frac{p}{q}}$$

for some $a \in L^q(\Omega)$ and $b > 0$.

Problem 4 (Banach fixed point, 4 points).
Consider the nonlinear boundary value problem

$$\begin{cases} -\Delta u + b(\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \in \mathbb{R}^n$ is a bounded, open set with $\partial\Omega$ of class C^2, $f \in L^2(\Omega)$, and $b : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous. Show that there is a unique weak solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$, provided the Lipschitz constant of b is small enough.

Total: 16 points