Nonlinear Partial Differential Equations I University of Bonn
Winter term 2016/2017 Prof. Dr. M. Disertori
Problem Sheet 8 (due Monday 19.12.2016) Dr. M. Bonacini

Problem 1 (Eigenvalues of symmetric, elliptic operators, 2+2 points).
Let $\Omega \subset \mathbb{R}^n$ be open, bounded, and connected. Consider the operator $Lu = -\sum_{i,j=1}^{n}(a_{ij}u_{x_j})_{x_i}$, where the coefficients $a_{ij} \in L^\infty(\Omega)$ are symmetric and uniformly elliptic, and the associated bilinear form

$$B(u,v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}u_{x_j}v_{x_i}\,\mathrm{d}x, \qquad u,v \in H_0^1(\Omega).$$

By the general theory, we have a sequence $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots$ of eigenvalues of the operator L (with Dirichlet boundary conditions), with the corresponding eigenfunctions $\{u_k\}_k$ forming an orthonormal basis of $L^2(\Omega)$.

a) Prove the formula

$$\lambda_k = \min\left\{ \frac{B(u,u)}{\|u\|_{L^2}^2} \,:\, u \in H_0^1(\Omega),\ (u,u_i)_{L^2} = 0 \text{ for } i = 1,\ldots,k-1 \right\},$$

where $(u,v)_{L^2}$ denotes the scalar product in $L^2(\Omega)$.

b) (*Courant minmax principle*) Prove that

$$\lambda_k = \sup_{\varphi_1,\ldots,\varphi_{k-1}\in L^2(\Omega)} \min\left\{ \frac{B(u,u)}{\|u\|_{L^2}^2} \,:\, u \in H_0^1(\Omega),\ (u,\varphi_i)_{L^2} = 0 \text{ for } i = 1,\ldots,k-1 \right\}.$$

Hint: choose $u = \sum_{i=1}^{k}\beta_i u_i$ for suitable β_i.

Problem 2 (Monotone dependence of eigenvalues on the domain, 4 points).
Let $a_{ij} \in L^\infty(\mathbb{R}^n)$ be symmetric and uniformly elliptic coefficients, and consider the elliptic operator $Lu := -\sum_{i,j=1}^{n}(a_{ij}u_{x_j})_{x_i}$. For any given open and bounded set $\Omega \subset \mathbb{R}^n$, denote by $\{\lambda_k(\Omega)\}_k$ the sequence of eigenvalues of L (with Dirichlet boundary conditions) on the domain Ω: that is, the values for which there exist nontrivial solutions to the boundary value problem

$$\begin{cases} Lu = \lambda_k(\Omega)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Show that if $\Omega_1 \subset \Omega_2$ are open and bounded, then $\lambda_k(\Omega_2) \le \lambda_k(\Omega_1)$ for every k.
Hint: you can first do the proof in the easiest case $k = 1$, using Rayleigh's formula; in the general case $k > 1$, use the two characterizations of λ_k given in Problem 1.

Please turn over.

Problem 3 (Mosco convergence and γ-convergence, 6 points).
Let D be a fixed open, bounded set. Let $\Omega_n \subset D$ be a sequence of open subsets of D, and let $\Omega \subset D$ be an open subset. We say that:

a) Ω_n *γ-converges to* Ω, if for every $f \in (H_0^1(D))'$ (the dual of $H_0^1(D)$), denoting by $u_f^{(n)}$ and u_f the weak solutions to the Dirichlet problems

$$\begin{cases} -\Delta u_f^{(n)} = f & \text{in } \Omega_n, \\ u_f^{(n)} = 0 & \text{on } \partial\Omega_n, \end{cases} \qquad \begin{cases} -\Delta u_f = f & \text{in } \Omega, \\ u_f = 0 & \text{on } \partial\Omega \end{cases}$$

respectively, then $u_f^{(n)}$ converges strongly to u_f in $H_0^1(D)$ (here, as usual, every function in $H_0^1(\Omega_n)$ is extended by zero outside Ω_n).

b) $H_0^1(\Omega_n)$ *converges to* $H_0^1(\Omega)$ *in the sense of Mosco*, if the following two conditions are satisfied:

(M1) for every $v \in H_0^1(\Omega)$, there exists a sequence $v_n \in H_0^1(\Omega_n)$ such that $v_n \to v$ strongly in $H_0^1(D)$;

(M2) for every subsequence $v_{n_k} \in H_0^1(\Omega_{n_k})$ which converges weakly in $H_0^1(D)$ to a function $v \in H_0^1(D)$, then $v \in H_0^1(\Omega)$.

Prove that Ω_n γ-converges to Ω if and only if $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco.
BONUS (extra credit: 4 points). Prove that, if Ω_n γ-converges to Ω, then all the eigenvalues of the Laplacian in the domain Ω_n converge to the corresponding eigenvalues in the domain Ω: $\lambda_k(\Omega_n) \to \lambda_k(\Omega)$ as $n \to \infty$, for every k (with the notation of Problem 2).
Hint: use the characterizations of eigenvalues given in Problem 1 to show the following implications:

$$\begin{aligned} &\text{(M1)} \quad \Longrightarrow \quad \limsup_{n\to\infty} \lambda_k(\Omega_n) \le \lambda_k(\Omega)\,, \\ &\text{(M2)} \quad \Longrightarrow \quad \lambda_k(\Omega) \le \liminf_{n\to\infty} \lambda_k(\Omega_n)\,. \end{aligned}$$

You might first try with the easiest case $k = 1$.

Problem 4 (Nodal domains, 2 points).
Let $\Omega \subset \mathbb{R}^n$ be open, bounded, and connected. Consider the operator $Lu = -\sum_{i,j=1}^n (a_{ij} u_{x_j})_{x_i}$, where the coefficients $a_{ij} \in C^\infty(\Omega)$ are symmetric and uniformly elliptic. Let u_k, $k \ge 1$, be the eigenfunctions of the operator L in Ω (with Dirichlet boundary conditions), associated to the eigenvalues λ_k. The connected components of the open sets

$$\Omega_k^+ = \{x \in \Omega \,:\, u_k(x) > 0\}\,, \qquad \Omega_k^- = \{x \in \Omega \,:\, u_k(x) < 0\}$$

are called the *nodal domains* of u_k.

Show that every eigenfunction u_k, $k \ge 2$, has to change sign in Ω. Moreover, let ω_k be one of the nodal domains of u_k. Show that the first eigenvalue of L in ω_k coincides with λ_k:

$$\lambda_1(\omega_k) = \lambda_k$$

(with the notation of Problem 2).

Total: 16 points, extra credit 4 points