Nonlinear Partial Differential Equations I University of Bonn
Winter term 2016/2017 Prof. Dr. M. Disertori
Problem Sheet 7 (due Monday 12.12.2016) Dr. M. Bonacini

Problem 1 (Uniqueness for various boundary conditions, 2+2+2 points).
Let $\Omega \subset \mathbb{R}^n$ be open, bounded, and connected, and suppose that Ω satisfies an interior ball condition at every point on $\partial\Omega$. If $x_0 \in \partial\Omega$, denote by $\nu(x_0)$ the exterior normal to an interior ball tangent to $\partial\Omega$ at x_0.
Consider $Lu := -\sum_{i,j=1}^n a_{ij}u_{x_i x_j} + \sum_{i=1}^n b_i u_{x_i} + cu$, where $a_{ij}, b_i, c \in C^0(\overline{\Omega})$, $c \geq 0$, and a_{ij} are uniformly elliptic. Assume that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution to $Lu = 0$ in Ω. Prove:

a) If the normal derivative $\frac{\partial u}{\partial \nu}$ is defined everywhere on $\partial\Omega$ and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, then u is constant in Ω. If furthermore $c > 0$ at some point in Ω, then $u \equiv 0$.

b) Assume that $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$, with $\partial_D\Omega \neq \emptyset$, and that $u \in C^1(\Omega \cup \partial_N\Omega)$ satisfies the mixed boundary condition

$$u = 0 \text{ on } \partial_D\Omega, \qquad \sum_{i=1}^n \beta_i(x)u_{x_i} = 0 \text{ on } \partial_N\Omega,$$

where $\beta(x) = (\beta_1(x), \dots, \beta_n(x))$ has a non-zero normal component (to the interior ball) at each point $x \in \partial_N\Omega$. Then $u \equiv 0$.

c) Assume that $u \in C^1(\overline{\Omega})$ satisfies the regular oblique derivative boundary condition

$$\alpha(x)u + \sum_{i=1}^n \beta_i(x)u_{x_i} = 0 \qquad \text{on } \partial\Omega,$$

where $(\beta \cdot \nu)\alpha > 0$ on $\partial\Omega$. Then $u \equiv 0$.

Problem 2 (Comparison principle for quasilinear elliptic equations, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Consider a second order quasilinear operator

$$Au := -\sum_{i,j=1}^n a_{ij}(x, \nabla u(x))u_{x_i x_j} + b(x, u(x), \nabla u(x))\,,$$

where the coefficients $a_{ij} \in C^0(\Omega \times \mathbb{R}^n)$ are symmetric and bounded,

$$\sum_{i,j=1}^n a_{ij}(x,\xi)\eta_i\eta_j \geq \theta|\eta|^2 \quad \text{for every } (x,\xi) \in \Omega \times \mathbb{R}^n$$

for some $\theta > 0$, and $b \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. Assume also that the maps $\xi \to a_{ij}(x,\xi)$, $\xi \to b(x,z,\xi)$ are continuously differentiable in $\mathbb{R}^n$ for every $(x,z) \in \Omega \times \mathbb{R}$, and b is non-decreasing in the second variable.

Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$Au \le Av \quad \text{in } \Omega, \qquad u \le v \quad \text{on } \partial\Omega.$$

Prove that $u \le v$ in Ω.
Hint: recall that for a continuously differentiable function $g : \mathbb{R}^n \to \mathbb{R}$ we have the mean value theorem

$$g(p) - g(q) = s \cdot (p - q), \qquad s = \int_0^1 \nabla g((1-t)q + tp)\,\mathrm{d}t\,.$$

Apply this property to the function

$$g(x, p) := -\sum_{i,j=1}^{n} a_{ij}(x, p) v_{x_i x_j}(x) + b(x, u(x), p)$$

to obtain

$$-\sum_{i,j=1}^{n} \Big(a_{ij}(x, \nabla u) - a_{ij}(x, \nabla v)\Big) v_{x_i x_j} + b(x, u, \nabla u) - b(x, u, \nabla v) = \beta(x) \cdot \nabla(u - v).$$

Then apply the weak maximum principle to the linear *operator*

$$Lw := -\sum_{i,j=1}^{n} a_{ij}(x, \nabla u(x)) w_{x_i x_j} + \beta(x) \cdot \nabla w(x)\,.$$

Problem 3 (Convexity of solutions to elliptic PDEs, 2+2+2 points).
Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and convex domain. Consider an elliptic operator $Lu := -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + f(u)$, where the *constant* coefficients a_{ij} are symmetric and elliptic, and

$$f \in C^1(\mathbb{R}), \quad f' > 0, \quad f \text{ concave.}$$

Suppose that $u \in C^2(\Omega)$ is a solution to $Lu = 0$ in Ω. Define, for $\lambda \in (0, 1)$, the concavity function

$$\mathcal{C}_\lambda(x, y) := u(\lambda x + (1 - \lambda)y) - \lambda u(x) - (1 - \lambda)u(y), \qquad (x, y) \in \Omega \times \Omega.$$

Notice that $\mathcal{C}_\lambda$ measures how much u fails to be convex; in particular, if $\mathcal{C}_\lambda \le 0$ in $\Omega \times \Omega$ for every $\lambda \in (0, 1)$, then u is convex in Ω.

a) Prove that $\mathcal{C}_\lambda$ cannot have a positive maximum at an interior point $(\bar{x}, \bar{y}) \in \Omega \times \Omega$. *Hint: argue by contradiction, and consider the function*

$$\bar{\mathcal{C}}_\lambda(\xi) := \mathcal{C}_\lambda(\bar{x} + \xi, \bar{y} + \xi),$$

which has a local maximum at $\xi = 0$. Also use the fact that, if $M = (m_{ij})_{ij}$ is a symmetric and negative semidefinite matrix, then $\sum_{i,j=1}^{n} a_{ij} m_{ij} \le 0$.

b) Generalize the previous result to the case in which $a_{ij} = a_{ij}(\nabla u)$, $f = f(u, \nabla u)$ depend also on ∇u:

$$a_{ij} \in C^0(\mathbb{R}^n), \qquad \sum_{ij=1}^{n} a_{ij}(\xi)\eta_i\eta_j \geq \theta|\eta|^2 \quad \text{for every } \xi, \eta \in \mathbb{R}^n,$$

$$f \in C^1(\mathbb{R} \times \mathbb{R}^n), \quad \frac{\partial f}{\partial u} > 0, \quad u \mapsto f(u, \xi) \text{ concave for every } \xi \in \mathbb{R}^n .$$

Hint: prove that $\nabla u(\bar{x}) = \nabla u(\bar{y}) = \nabla u(\lambda\bar{x} + (1-\lambda)\bar{y})$ *at an interior maximum point* $(\bar{x}, \bar{y}) \in \Omega \times \Omega$ *of* $\mathcal{C}_\lambda$*. Then repeat the proof as in the previous case.*

c) Let Ω be strictly convex (that is, $\lambda x + (1-\lambda)y \in \Omega$ for every $x, y \in \overline{\Omega}$ and $\lambda \in (0,1)$). Assume that $u(x) \to \infty$ uniformly as $\operatorname{dist}(x, \partial\Omega) \to 0$. Prove that u is convex in Ω.

Total: 16 points