Nonlinear Partial Differential Equations I | University of Bonn
Winter term 2016/2017 | Prof. Dr. M. Disertori
Problem Sheet 6 (due Monday 05.12.2016) | Dr. M. Bonacini

Problem 1 (Weak formulation of the maximum principle, 3+1+2 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and consider the elliptic operator $Lu = -\sum_{i,j=1}^n \big(a_{ij}u_{x_i}\big)_{x_j}$, where the coefficients $a_{ij} \in L^\infty(\Omega)$ are symmetric and uniformly elliptic. Assume that $u \in H^1(\Omega)$ is a subsolution for the operator L, that is $Lu \le 0$ in the sense

$$\sum_{i,j=1}^n \int_\Omega a_{ij}u_{x_i}\varphi_{x_j}\,\mathrm{d}x \le 0 \qquad \text{for every } \varphi \in H_0^1(\Omega) \text{ with } \varphi \ge 0.$$

a) Prove that, if $(u-M)^+ \in H_0^1(\Omega)$, then $u \le M$ almost everywhere in Ω.
Hint: recall Problem 2 in Exercise Sheet 1.

b) (Comparison principle) Deduce that, if $u, v \in H^1(\Omega)$ satisfy $Lu \le 0$, $Lv \ge 0$, and $(u-v)^+ \in H_0^1(\Omega)$, then $u \le v$ almost everywhere in Ω.

c) If $\partial\Omega$ is of class C^1, show that the condition $(u-M)^+ \in H_0^1(\Omega)$ is equivalent to $Tu \le M$ almost everywhere on $\partial\Omega$, where Tu denotes the trace of u. (The advantage of writing $(u-M)^+ \in H_0^1(\Omega)$ is that you don't need any regularity of $\partial\Omega$, while the notion of trace requires at least the Lipschitz regularity of the boundary!)

Problem 2 (A priori bounds for inhomogeneous equations, 2+2+2 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and consider $Lu := -\sum_{i,j=1}^n a_{ij}u_{x_ix_j} + \sum_{i=1}^n b_iu_{x_i} + cu$, where $a_{ij}, b_i, c \in C^0(\overline{\Omega})$, and $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \theta|\xi|^2$ for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$, where $\theta > 0$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $Lu = f$ for some $f \in C^0(\overline{\Omega})$.

a) Assume also that $c \ge 0$. Show that

$$\sup_\Omega |u| \le \sup_{\partial\Omega} |u| + C \sup_\Omega |f|\,,$$

where $C > 0$ is a constant depending only on θ, $\|b\|_\infty$, and on the diameter of Ω.
Hint: assume without loss of generality that $\Omega \subset \{x \in \mathbb{R}^n : 0 < x_1 < d\}$ *for* $d = \operatorname{diam}\Omega$. *Consider the function* $v(x) = \sup_{\partial\Omega}|u| + \frac{1}{\theta}(e^{\alpha d} - e^{\alpha x_1})\sup_\Omega |f|$ *for some* $\alpha > 0$ *sufficiently large.*

b) Let C be the constant found in the previous step. Assume that

$$C_1 := 1 - C\sup_\Omega |c^-| > 0.$$

Show that

$$\sup_\Omega |u| \le \frac{1}{C_1}\left(\sup_{\partial\Omega}|u| + C\sup_\Omega |f|\right).$$

c) Prove uniqueness in $C^2(\Omega) \cap C^0(\overline{\Omega})$ of solutions to the Dirichlet problem $Lu = f$ in Ω, $u = g$ on $\partial\Omega$, $g \in C^0(\partial\Omega)$, provided that the domain Ω is sufficiently narrow.

Problem 3 (Comparison with Perron's solution, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $w \in H^1(\Omega) \cap C^0(\overline{\Omega})$. Let $u \in H^1(\Omega)$ be the unique weak solution to the boundary value problem

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u - w \in H^1_0(\Omega). \end{cases} \tag{1}$$

The aim of the exercise is to prove the following characterization of u:

(i) for every function $v \in C^0(\overline{\Omega})$ superharmonic[1] in Ω and such that $v(x) \geq w(x)$ for every $x \in \partial\Omega$, one has $u(y) \leq v(y)$ for every $y \in \Omega$;

(ii) for every function $v \in C^0(\overline{\Omega})$ subharmonic in Ω and such that $v(x) \leq w(x)$ for every $x \in \partial\Omega$, one has $u(y) \geq v(y)$ for every $y \in \Omega$.

This means that the function u coincides with the solution to the boundary value problem (1) constructed with *Perron's method* (see, for instance, [D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*]).
Hint: use Problem 4 below, together with the result of Problem 4 in Exercise Sheet 5.

Problem 4 (Superharmonic functions, extra credit: 4 points).
Prove that every bounded, superharmonic function v belongs to $H^1_{\mathrm{loc}}(\Omega)$. If you drop the assumption that v is bounded, can you exhibit a counterexample?
Hint: without loss of generality we can assume that $0 \leq v \leq M$. Consider the convolutions $v_\varepsilon = v * \rho_\varepsilon$, where ρ_ε is a standard sequence of mollifiers, for which we have $-\Delta v_\varepsilon \geq 0$. Fix also $\Omega' \subset \Omega$ with $\overline{\Omega'} \subset \Omega$, and a cut-off function $\varphi \in C^\infty_c(\Omega)$, $0 \leq \varphi \leq 1$, $\varphi = 1$ on Ω'. By using the test function $v_\varepsilon \varphi^2$, prove that the L^2-norms of ∇v_ε are uniformly bounded in Ω'.

Total: 16 points, extra credit 4 points

Extra credit counts towards your personal score but not towards the total marks.

[1] A function $v \in C^0(\Omega)$ is superharmonic in Ω if $-\Delta v \geq 0$ in the distributional sense:

$$(-\Delta v, \varphi) := -\int_\Omega v \Delta\varphi \, \mathrm{d}x \geq 0 \qquad \text{for every } \varphi \in C^\infty_c(\Omega),\ \varphi \geq 0.$$

A function $v \in C^0(\Omega)$ is subharmonic in Ω if $-v$ is superharmonic in Ω.