Nonlinear Partial Differential Equations I University of Bonn
Winter term 2016/2017 Prof. Dr. M. Disertori
Problem Sheet 5 (due Monday 28.11.2016) Dr. M. Bonacini

Problem 1 (Boundary $\mathbf{H^2}$ regularity, 4 points).
Let $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ and let $B_r^+ = B_r \cap \{x_n > 0\}$ be the upper half ball. Assume that $a_{ij} \in C^1(\overline{B}_1^+)$ are symmetric and uniformly elliptic coefficients, $b_i, c \in L^\infty(B_1^+)$, and $f \in L^2(B_1^+)$. Let $u \in H_0^1(B_1^+)$ be a weak solution to the elliptic equation

$$Lu := -\sum_{i,j=1}^n \big(a_{ij}u_{x_i}\big)_{x_j} + \sum_{i=1}^n b_i u_{x_i} + cu = f \qquad \text{in } B_1^+ .$$

Prove that $u \in H^2(B_r^+)$ for every $r \in (0,1)$, with

$$\|u\|_{H^2(B_r^+)} \le C\big(\|f\|_{L^2(B_1^+)} + \|u\|_{L^2(B_1^+)}\big)$$

for a constant C depending only on r and on the coefficients of L.
Hint: Argue as in the proof of the interior regularity theorem, using the test functions

$$v := -D_k^{-h}\big(\zeta^2 D_k^h u\big) \qquad k \in \{1,\dots,n-1\},\ |h| \text{ small},$$

where ζ is a suitable cut-off function. Obtain an estimate on $\|u_{x_i x_j}\|_{L^2}$ for $i \in \{1,\dots,n-1\}$, $j \in \{1,\dots,n\}$. Then find an estimate also on $u_{x_n x_n}$ by using the equation.

Problem 2 (Regularity for a semilinear problem, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Consider an elliptic operator $Lu = -\sum_{i,j=1}^n (a_{ij}u_{x_j})_{x_i}$, where the coefficients $a_{ij} \in C^2(\Omega)$ are symmetric and uniformly elliptic.

a) Suppose that $f \in C^1(\mathbb{R})$ satisfies $\|f'\|_\infty < \infty$. Assume that $u \in H^1(\Omega)$ is a weak solution to
$$Lu = f(u) \quad \text{in } \Omega. \tag{1}$$
Show that $u \in H^3_{\mathrm{loc}}(\Omega)$.

b) Assume further that $a_{ij} \in C^3(\Omega)$ and $f \in C^2(\Omega)$ with $\|f''\|_\infty < \infty$. Prove that $u \in H^4_{\mathrm{loc}}(\Omega)$, provided that the dimension n of the space is not too large.

c) Let $f(u) = |u|^p$, $p \ge 1$. For which values of p can you write a weak formulation of the equation (1) in $H_0^1(\Omega)$? For which values of p can you ensure that a weak solution u to (1) belongs to $H^2_{\mathrm{loc}}(\Omega)$?

Please turn over.

Problem 3 (Regularity in a domain with corner, 4 points).
In $\mathbb{R}^2$ use the polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ and define the angular domain

$$\Omega := \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \,:\, r \in (0,1),\, \theta \in (0,\omega)\},$$

for $\omega \in (0, 2\pi)$.

a) Check that the function $u(x,y) = r^{\frac{\pi}{\omega}} \sin(\frac{\pi}{\omega}\theta)$ lies in $H^1(\Omega)$ and solves

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial_D\Omega := \{(r\cos\theta, r\sin\theta) \,:\, 0 \le r \le 1,\, \theta \in \{0,\omega\}\}, \\ \nabla u \cdot \nu = \frac{\pi}{\omega}\sin(\frac{\pi}{\omega}\theta) & \text{on } \partial_N\Omega := \partial\Omega \backslash \partial_D\Omega. \end{cases}$$

For which values of ω do we have $u \in H^2(\Omega)$?

b) For those values of ω such that $u \notin H^2(\Omega)$, find a function $f \in C^0(\overline{\Omega})$ such that the unique solution $w \in H^1_0(\Omega)$ of the Dirichlet problem $\Delta w = f$ in Ω lies in $H^1(\Omega)$ but not in $H^2(\Omega)$.

Problem 4 (Convergence of weak solutions, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $\Omega_k \subset \Omega$ be an increasing sequence of open subsets of Ω such that $\Omega = \cup_{k=1}^{\infty}\Omega_k$. Let $w \in H^1(\Omega)$ and let $u_k \in H^1(\Omega_k)$ be the unique weak solution to

$$\begin{cases} Lu_k = 0 \text{ in } \Omega_k, \\ u_k - w \in H^1_0(\Omega_k), \end{cases}$$

where $Lu = -\sum_{i,j=1}^n (a_{ij}u_{x_j})_{x_i}$ and the coefficients $a_{ij} \in L^\infty(\Omega)$ are uniformly elliptic. By setting

$$\tilde{u}_k(x) := \begin{cases} u_k & \text{in } \Omega_k, \\ w & \text{in } \Omega\backslash\Omega_k, \end{cases}$$

show that $\tilde{u}_k$ converges strongly in $H^1(\Omega)$ to the unique weak solution u to

$$\begin{cases} Lu = 0 \text{ in } \Omega, \\ u - w \in H^1_0(\Omega). \end{cases}$$

Total: 16 points