Nonlinear Partial Differential Equations I | University of Bonn
Winter term 2016/2017 | Prof. Dr. M. Disertori
Problem Sheet 4 (due Monday 21.11.2016) | Dr. M. Bonacini

Problem 1 (Neumann boundary conditions, 2+2 points).
Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected set with C^1 boundary, and consider the elliptic operator

$$Lu := -\sum_{i,j=1}^{n} D_{x_i}\big(a_{ij}(x)D_{x_j}u(x)\big)$$

where the coefficients $a_{ij} \in L^\infty(\Omega)$ are symmetric and uniformly elliptic.

a) Show that, for every $f \in L^2(\Omega)$ with $\int_\Omega f(x)\,\mathrm{d}x = 0$, there exists a weak solution $u \in H^1(\Omega)$ of the problem $Lu = f$, in the sense that

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij} D_{x_j}u D_{x_i}v \,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x \qquad \text{for every } v \in H^1(\Omega),$$

and that such solution is unique up to constants (that is, the difference of any two solutions is a constant).

b) Assume that $a_{ij}, u \in C^\infty(\overline{\Omega})$. Which boundary condition does u satisfy? How does this condition become in the case of the Laplace operator $Lu = -\Delta u$?

Problem 2 (Harmonic reflection, 4 points).
Denote a generic point $x \in \mathbb{R}^n$ by $x = (\bar{x}, x_n)$, with $\bar{x} \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Let $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball in $\mathbb{R}^n$, and let $B^+ = B_1 \cap \{x : x_n > 0\}$ be the upper half ball. Assume that $u \in H^1(B^+)$ satisfies

$$\int_{B^+} \nabla u \cdot \nabla\varphi \,\mathrm{d}x = 0 \qquad \text{for every } \varphi \in C_c^\infty(B_1).$$

Show that the function

$$\tilde{u}(\bar{x}, x_n) := \begin{cases} u(\bar{x}, x_n) & \text{if } x_n > 0, \\ u(\bar{x}, -x_n) & \text{if } x_n < 0 \end{cases}$$

belongs to $H^1(B_1)$ and is harmonic in B_1.

Please turn over.

Problem 3 (Biharmonic equation, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open, bounded with smooth boundary. Consider the following boundary value problem for the biharmonic equation

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta^2 u = \sum_{i,j=1}^n \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2}$. Formulate a suitable weak formulation of the problem in the space $H_0^2(\Omega)$ and prove existence and uniqueness of a weak solution for every $f \in L^2(\Omega)$.

Problem 4 (Galerkin method and Céa's Lemma, 2+2 points).
Let H be a real Hilbert space, $F : H \to \mathbb{R}$ a linear and continuous functional, and let $B : H \times H \to \mathbb{R}$ be a symmetric, bounded and coercive bilinear form:

$$B(u,v) \leq C\|u\|\|v\|, \qquad B(u,u) \geq \alpha\|u\|^2 \qquad \text{for every } u, v \in H.$$

Let (H_n) be a sequence of finite dimensional subspaces such that $H_n \subset H_{n+1}$ for every n and $\overline{\cup_{n=1}^\infty H_n} = H$.

a) Show that for every n the equation

$$B(u,v) = F(v) \qquad \text{for every } v \in H_n$$

has a unique solution $u_n \in H_n$. Moreover, show that the sequence $(u_n)_n$ converges weakly in H to the unique solution u to

$$B(u,v) = F(v) \qquad \text{for every } v \in H.$$

b) Prove the following estimate for the error between the real solution u and the approximate solution u_n:

$$\|u_n - u\| \leq \frac{C}{\alpha} \inf\{\|u - v_n\| : v_n \in H_n\}$$

(u_n is, up to a constant, as close to u as the best approximation in H_n).
Hint: show that $u_n - u$ is orthogonal to H_n with respect to B.

Total: 16 points