Nonlinear Partial Differential Equations I — University of Bonn
Winter term 2016/2017 — Prof. Dr. M. Disertori
Problem Sheet 3 (due Monday 14.11.2016) — Dr. M. Bonacini

Problem 1 (Approximate point spectrum, 4 points).
Let H be an Hilbert space and let $T : H \to H$ be a bounded linear operator. Assume that T is symmetric.

a) Suppose that $T - \lambda I$ is injective. Show that $R(T - \lambda I)$ is dense in H (where $R(T - \lambda I)$ denotes the range of $T - \lambda I$).
Hint: if not, there would exist $x \in H$ orthogonal to the closure of $R(T - \lambda I)$.

b) Suppose that $R(T - \lambda I)$ is dense in H and that

$$\inf_{\|x\|=1} \|Tx - \lambda x\| > 0 .$$

Show that $R(T - \lambda I) = H$.

c) Denote by $\sigma(T)$ the *spectrum* of T. Prove that $\lambda \in \sigma(T)$ if and only if there exists a sequence $x_n \in H$, $\|x_n\| = 1$, for which

$$\lim_{n\to\infty} \|Tx_n - \lambda x_n\| = 0 .$$

(Thus every $\lambda \in \sigma(T)$ which is not an eigenvalue of T is an "approximate" eigenvalue).

Problem 2 (Spectral properties of the shift, 4 points).
Let $E = \ell^2$ be the space of the sequences $x = (x_1, x_2, \ldots, x_n, \ldots)$ such that

$$\|x\|_2 := \left(\sum_{k=1}^{\infty} |x_k|^2 \right)^{\frac{1}{2}} < \infty .$$

Consider the bounded, linear operators $S_r, S_l : \ell^2 \to \ell^2$ defined by

$$S_r x = (0, x_1, x_2, \ldots, x_{n-1}, \ldots), \qquad S_l x = (x_2, x_3, x_4, \ldots, x_{n+1}, \ldots)$$

for $x \in \ell^2$.

a) Determine $\|S_l\|$ and $\|S_r\|$. Is S_r or S_l a compact operator?

b) Prove that $\sigma(S_r) = [-1, 1]$ and $\sigma_p(S_r) = \emptyset$.

c) Prove that $\sigma(S_l) = [-1, 1]$ and $\sigma_p(S_l) = (-1, 1)$.

d) Determine the adjoint operators S_r^*, S_l^*.

Please turn over.

Problem 3 (Volterra operator, 4 points).
Let $X = L^2(0,1)$ and define the operator $V : X \to X$ by

$$Vu(x) = \int_0^x u(t)\,\mathrm{d}t\,.$$

Arguing as in Problem 2 in Problem Sheet 2, you can check that V is compact (you don't have to prove it again).

a) Determine the spectrum $\sigma(V)$ and the point spectum $\sigma_p(V)$.

b) Give an explicit formula for $(V - \lambda I)^{-1}$ when $\lambda \in \rho(V)$ (the resolvent set of V).

c) Determine the adjoint V^*.

Problem 4 (Weak solutions as minimizers, 4 points).
Let H be an Hilbert space, let $F : H \to R$ be a linear functional, and let $B : H \times H \to \mathbb{R}$ be a symmetric, bilinear form, such that

$$B(u,u) \geq 0 \qquad \text{for every } u \in H.$$

Show that u satisfies

$$B(u,v) = F(v) \qquad \text{for every } v \in H$$

if and only if u is minimizer of the problem

$$\min\left\{ \frac{1}{2}B(v,v) - F(v) \,:\, v \in H \right\}.$$

Deduce that every weak solution $u \in H_0^1(\Omega)$ to

$$-\sum_{i,j=1}^{n} D_{x_i}\big(a_{ij} D_{x_j} u\big) = f$$

(with $a_{ij} \in L^\infty(\Omega)$ symmetric and elliptic, $f \in L^2(\Omega)$) is a minimizer of some functional.

Total: 16 points