Nonlinear Partial Differential Equations I | University of Bonn
Winter term 2016/2017 | Prof. Dr. M. Disertori
Problem Sheet 2 (due Monday 07.11.2016) | Dr. M. Bonacini

Problem 1 (Interpolation inequality, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with C^1 boundary. Show that for every $\varepsilon > 0$ there exists a constant C_ε such that

$$\|Du\|_{L^2(\Omega)} \leq \varepsilon \|D^2 u\|_{L^2(\Omega)} + C_\varepsilon \|u\|_{L^2(\Omega)}$$

for every $u \in W^{2,2}(\Omega)$. Recall that $\|D^2 u\|^2_{L^2(\Omega)} = \sum_{\alpha,\beta=1}^n \|\partial_\alpha \partial_\beta u\|^2_{L^2(\Omega)}$.
Hint: argue by contradiction, as in the proof of Poincaré's inequality; in particular, apply the compact Sobolev embedding to the first derivatives to extract a suitable subsequence.

Problem 2 (Integration operator, 4 points).
Let $E = C([0,1])$ with the uniform norm, and let $T : E \to E$ be the operator defined by

$$Tf(x) = \int_0^x f(y)\,\mathrm{d}y$$

for $f \in E$ and $x \in [0,1]$. Check that T is compact and prove that $T(B_E)$ is not closed, where $B_E := \{f \in E : \|f\| \leq 1\}$.
Hint: apply Ascoli-Arzelà Theorem.

Problem 3 (Approximation of compact operators by finite-rank operators, 2+2 points).
Let X, Y be Banach spaces, and let $T : X \to Y$ be a bounded, linear operator.

a) Let $T_n : X \to Y$ be bounded, linear operators such that $\|T - T_n\| \to 0$, and assume that each T_n has finite-dimensional range. Prove that T is compact.

b) Let $\Omega \subset \mathbb{R}^n$ be open, let $p \in (1,\infty)$ and $q = \frac{p}{p-1}$ be conjugate exponents, and let $K \in L^p(\Omega \times \Omega)$. Define the operator $T_K : L^q(\Omega) \to L^p(\Omega)$ by setting

$$T_K f(x) = \int_\Omega K(x,y) f(y)\,\mathrm{d}y$$

for $f \in L^q(\Omega)$ and $x \in \Omega$. Prove that T_K is well-defined and is a compact operator.
Hint: you can use (without proving it) the fact that K is the limit in $L^p(\Omega \times \Omega)$ of a sequence of kernels K_n of the form

$$K_n(x,y) = \sum_{i=1}^k g_i(x) h_i(y)$$

for functions $g_i, h_i \in L^p(\Omega)$.

Please turn over.

Problem 4 (Normal operators, 2+2 points).
Let H be a real Hilbert space. A bounded, linear operator T on H is said to be *normal* if $TT^* = T^*T$, where T^* denotes the adjoint of T.

a) Show that T is normal if and only if $\|Tx\| = \|T^*x\|$ for every $x \in H$.

b) Let T be a normal operator. Show that for every $\lambda \in \mathbb{R}$

$$Ker(T - \lambda I) = Ker(T^* - \lambda I)\,.$$

(In particular, T and T^* have the same eigenvalues.)

Hint: observe that $T - \lambda I$ is normal too.

Total: 16 points