Nonlinear Partial Differential Equations I University of Bonn
Winter term 2016/2017 Prof. Dr. M. Disertori
Problem Sheet 1 (due Monday 31.10.2016)

Problem 1 (Chain rule, 3 points).
Let $\Omega \subset \mathbb{R}^n$ be an open set, $p \in [1, \infty)$. Let $f : \mathbb{R} \to \mathbb{R}$ be of class C^1, with f' bounded, and assume that $f(0) = 0$ if Ω has infinite measure. Show that, for every $u \in W^{1,p}(\Omega)$, $f \circ u \in W^{1,p}(\Omega)$ and $D(f \circ u) = f'(u)Du$.
Hint: approximate u in $W^{1,p}_{\mathrm{loc}}(\Omega)$ by a sequence $u_k \in C^\infty(\Omega)$ of smooth functions.

Problem 2 (Stampacchia's Lemma, 2+1 points).
Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in W^{1,p}(\Omega)$, $p \in [1, \infty)$.

a) Show that $u^+, u^-, |u| \in W^{1,p}(\Omega)$, where $u^+ := \max\{u, 0\}$, $u^- := \min\{u, 0\}$, and that

$$Du^+(x) = \begin{cases} Du(x) & \text{if } u(x) > 0, \\ 0 & \text{if } u(x) \le 0, \end{cases}$$
$$Du^-(x) = \begin{cases} 0 & \text{if } u(x) \ge 0, \\ Du(x) & \text{if } u(x) < 0. \end{cases}$$

b) Show that $Du = 0$ almost everywhere on the set $\{x \in \Omega : u(x) = c\}$, where $c \in \mathbb{R}$ is any given constant.

Hint: use the result in Problem 1 with the sequence of functions

$$f_\varepsilon(t) := \begin{cases} \sqrt{t^2 + \varepsilon^2} - \varepsilon & \textit{if } t > 0, \\ 0 & \textit{if } t \le 0. \end{cases}$$

Problem 3 (Integration by parts, 3 points).
Let $\Omega \subset \mathbb{R}^n$ be open, bounded with C^1 boundary. Show that for every $u \in W^{1,p}(\Omega)$ and $\varphi \in C^1(\overline{\Omega})$ one has

$$\int_\Omega u D\varphi \, \mathrm{d}x = -\int_\Omega \varphi Du \, \mathrm{d}x + \int_{\partial\Omega} T(u)\varphi\nu \, \mathrm{d}\mathcal{H}^{n-1} \,,$$

where ν denotes the exterior unit normal to $\partial\Omega$, and $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is the trace operator.
Hint: remember that $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$.

Problem 4 (Traces of Sobolev functions, 3 points).
Let $\Omega \subset \mathbb{R}^n$ be open, bounded with C^1 boundary. Let $u \in W^{1,p}(\Omega)$ and $v \in W^{1,p}(\mathbb{R}^n \backslash \overline{\Omega})$, $1 < p < \infty$. Prove that the function

$$w(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ v(x) & \text{if } x \in \mathbb{R}^n \backslash \overline{\Omega} \end{cases}$$

belongs to $W^{1,p}(\mathbb{R}^n)$ if and only if $T(u) = T(v)$, where T is the trace operator on $\partial\Omega$.

Problem 5 (The space $\mathbf{W_0^{1,p}}$, 2+2 points).
Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with C^1 boundary, and let $1 < p < \infty$. For $u \in L^p(\Omega)$ define

$$\bar{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

a) Show that $u \in W_0^{1,p}(\Omega)$ if and only if $\bar{u} \in W^{1,p}(\mathbb{R}^n)$.

b) Show that $u \in W_0^{1,p}(\Omega)$ if and only if there exists a constant C such that

$$\left| \int_{\mathbb{R}^n} \bar{u} D\varphi \, \mathrm{d}x \right| \leq C \|\varphi\|_{L^{p'}(\mathbb{R}^n)}$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Hint: use Riesz Representation Theorem.

Total: 16 points