Gamma convergence for phase transitions in impenetrable elastic materials

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Abstract: We study the family of functionals

$$I_\varepsilon[u] = \int_\Omega \frac{1}{\varepsilon} W(\nabla u) + \varepsilon |\nabla^2 u|^2 \, dx,$$

with $u : \Omega \subset \mathbf{R}^n \to \mathbf{R}^n$ representing the deformation of an elastic body, and W the energy density, which vanishes for all matrices in $K = SO(n)A \cup SO(n)B$. The energy I_ε describes an elastic material with two preferred gradients and surface tension, the so-called two-well problem of solid-solid phase transitions. The Gamma limit of the functionals I_ε was determined, for $n = 2$, in [5], the crucial step in the proof is to derive rigidity estimates in order to control the local rotations of minimizing sequences. While [5] treats the case that W has quadratic growth at infinity, we treat here the case that W does not permit self-penetration, i.e. $W(F) = \infty$ for $\det F < 0$. We restrict to $n = 2$ and exploit results of [5].

1. Introduction.

Our investigations follow in spirit the pioneering work of Modica and Mortola who studied in [9] functionals of the type

$$J_\varepsilon[v] = \int_\Omega \frac{1}{\varepsilon} W(v) + \varepsilon |\nabla v|^2 \, dx, \tag{1}$$

with $W \geq 0$, $W(\xi) = 0$ iff $\xi \in \{a, b\}$. They found that sequences v^ε with bounded energies $J_\varepsilon[v^\varepsilon]$ are precompact and that limits have a particular form, namely $u_0(x) = a\chi_E + b(1-\chi_E)$, where χ_E is the characteristic function of a set E with bounded perimeter. Moreover, the (minimal) limiting energy of sequences $u^\varepsilon \to u_0$ can be characterized and is proportional to the perimeter of E. The precise statement is that the Gamma limit of the functionals J_ε is given by $J_0[u_0] = k\,\mathrm{Per}_\Omega(E)$ for some $k \in \mathbf{R}$, which can be explicitly characterized as the integral of $W^{1/2}(t)$ over $t \in [a, b]$.

In the theory of solid-solid phase transitions one is interested in *deformations* $u : \Omega \to \mathbf{R}^n$, and the free energy associated to a deformation depends not on the local value of u, but rather on the strain, i.e. on ∇u. We are therefore interested in a free energy $W(\nabla u)$ with $W : \mathbf{R}^{n\times n} \to \mathbf{R}$, and the effect of the (small) surface tension is to penalize second derivatives. We therefore investigate

$$I_\varepsilon[u, \Omega] = \int_\Omega \frac{1}{\varepsilon} W(\nabla u) + \varepsilon |\nabla^2 u|^2 \, dx. \tag{2}$$

The Gamma limit of these functionals for $W(F) = 0$ iff $F \in \{A, B\}$ was derived in [3]. There are some principal new effects in the gradient theory. One is that the matrices A and B must be rank-one connected in order to have a nontrivial limit, say $A - B = a \otimes \nu$. Limits u_0 of sequences u_ε with bounded energy then have the special structure $\nabla u_0 = A\,\chi_E + B\,(1 - \chi_E)$, and the boundary of E consists of (subsets of) hyperplanes with normal ν. Despite this simplification of the set of possible limits, the result is less sharp than that for J_ε in the sense that the limiting surface energy k could only be characterized implicitly as

$$k = \inf \left\{ \liminf_{i\to\infty} I_{\varepsilon_i}[u_i, Q_\nu] : \varepsilon_i \to 0, u_i \to u_0^\nu \text{ in } L^1 \right\}. \tag{3}$$

Here, Q_ν is a unit square centered in the origin with one side parallel to ν and u_0^ν is a continuous function with $\nabla u_0^\nu(x) = A$ if $x\cdot\nu > 0$, and $\nabla u_0^\nu(x) = B$ if $x\cdot\nu < 0$. Corresponding to this more complex characterization of k, it was shown with an explicit example that the optimale profile in (3) need not be one-dimensional [3].

With the above definition of k the proof of the $\Gamma - \liminf$ is obtained rather directly with scaling and covering arguments; the intricate point is in the constructive part of the $\Gamma - \limsup$, where, in the case that more than one interface is present in the limit, the optimal profiles of (3) must be glued together. This problem is solved in [3] by a two-fold approximation, 'of the gradient' and 'of the function' on different length scales, in order to bridge from a low-energy profile to an affine deformation.

The next step was to allow for free energies W that are consistent with the requirement of frame-indifference in elastic materials. Our contributions in [4] and [5] led to an extension of the above Gamma-limit result, for the case $n = 2$, to $SO(2)$-invariant free energies W, that is for W satisfying $W(QF) = W(F)$ for all $Q \in SO(2)$. The key difficulty is now to obtain a control of the rotations. Loosely speaking, a deformation u_ε with low energy $I_\varepsilon(u_\varepsilon)$ satisfies

$$\nabla u_\varepsilon(x) = Q(x)A\,\chi_E(x) + Q(x)B\,(1 - \chi_E(x)) + o(\sqrt{\varepsilon}) \tag{4}$$

in the L^2-sense. In the proof of the $\Gamma - \limsup$ inequality we have to start from optimal profiles u_ε as in (3), and match these low-energy functions u_ε with an affine function. In particular, we have to find that the set E in (4) is large, and that the field of rotations $Q(.)$ is almost constant. This is done by the rigidity estimates in [4] and [5].

Here we consider a model which does not permit (local) interpenetration of matter, i.e. an energy density W which is infinite for deformations gradients with negative determinant.

This introduces a nonconvex constraint which generates new difficulties in the explicit construction, while at the same time the proof of the rigidity estimate is somewhat simplified. The new construction is presented in Proposition 2, which constitutes the main new result of this paper. In Proposition 1 we give a simplified proof of the central rigidity estimate which exploits the non-interpenetrability assumption.

2. Main result. The result presented here differs from that of [5] in its assumptions on W. As in [5] we assume that

$$W : \mathbf{R}^{2\times 2} \to [0,\infty] \text{ satisfies } W(QF) = W(F) \text{ for all } F \in \mathbf{R}^{2\times 2}, Q \in SO(2), \tag{5}$$

$$W \text{ vanishes on } K = SO(2)\{A, B\}, \tag{6}$$

with fixed matrices $A, B \in \mathbf{R}^{2\times 2}$, rank-one connected and such that $AB^{-1} \notin SO(2)$, since otherwise we are in the case of only one well. In this contribution we study the growth assumption

$$W \text{ continuous on } \{F \in \mathbf{R}^{2\times 2} | \det F > 0\}, \tag{7}$$

$$W(F) = +\infty \text{ if } \det F \le 0, \quad W(F) \ge c_1 \operatorname{dist}^2(F, K), \tag{8}$$

$$W(F) \le c_2 \operatorname{dist}^2(F, K) \text{ for all } F \text{ in a neighborhood of } K, \tag{9}$$

with constants $c_1, c_2 > 0$.

As in [3,4,5] we define $k(\nu)$ by

$$k(\nu) = \inf\left\{ \liminf_{i\to\infty} I_{\varepsilon_i}[u_i, Q_\nu] : \varepsilon_i \to 0, u_i \to u_0^\nu \text{ in } L^1 \right\}, \tag{10}$$

and note that it is positive and satisfies $k(\nu) = k(-\nu)$. Here, Q_ν is a unit square centered in the origin with one side parallel to ν and u_0^ν is a continuous function with $\nabla u_0^\nu(x) = A$ if $x \cdot \nu > 0$, and $\nabla u_0^\nu(x) = QB$ if $x \cdot \nu < 0$, $Q \in SO(2)$ being such that $A - QB = a \otimes \nu$ for some $a \in \mathbf{R}^2$. We shall prove that the limit functional is finite only on functions u such that ∇u takes only values in K, and on such functions it is proportional to the length of the interface between the region where $\nabla u \in SO(2)A$ and the one where $\nabla u \in SO(2)B$.

$$I_0[u, \Omega] = \begin{cases} \int_{J_{\nabla u}} k(\nu) d\mathcal{H}^1 & \text{if } \nabla u \in BV(\Omega, K), \\ +\infty & \text{else,} \end{cases} \tag{11}$$

where $J_{\nabla u}$ denotes the jump set of ∇u and ν the normal to it. Dolzmann and Müller [7] have characterized the functions u that appear in the first case as local laminates that are locally affine and have jumps only between the A and the B region. The jump set consists locally of segments that are orthogonal to one of the (at most) two possible normal vectors ν determined by A and B.

Theorem. *Let $\Omega \subset \mathbf{R}^2$ be a strictly star-shaped, bounded Lipschitz domain and let W satisfy (5)– (9). Then*

$$\Gamma - \lim_{\varepsilon\to 0} I_\varepsilon = I_0$$

with respect to the strong L^1 topology.

We call an open set Ω strictly star-shaped if there is a point $x \in \Omega$ such that for any $y \in \partial\Omega$ the segment (x, y) is contained in Ω.

The $\Gamma - \liminf$ inequality follows from the one of [5], which in turn was derived, following [3,4], combining a compactness result and the definition of $k(\nu)$ of (10).

We therefore focus on the proof of the $\Gamma - \limsup$. In this part of the proof we are given u_0 with several interfaces and have to construct sequences u_ε that converge to u_0 in L^1 with optimal energy. In the case that u_0 has only a single interface, we can use the sequence u_ε^{BA} that appears in (10) (but we still need to show that the result does not depend on the sequence $\varepsilon_i \to 0$, see [4]). The intricate part is to deal with u_0 with more than one interface. Then, in the construction of a low energy sequence, we can again use locally around each interface the sequence u_ε^{BA} (or its rotated version $u_\varepsilon^{AB}(\cdot) = -u^{BA}(-\cdot)$, but between the interfaces we have to match the two approximations. This construction can be done if we can modify the functions u_ε^{BA} so that they become affine at some distance from the interface. Precisely this statement is verified in Proposition 2, which is the main result of this contribution.

Note that we formulate Proposition 2 for low energy maps u. In the proof of the theorem we apply the proposition to the restriction of the maps u_ε^{BA} to the part $D = \{x : x\cdot\nu > 1/2\}$. Since the energy of u_ε^{BA} is concentrated along the boundary, the energy in the part D of the square is small. Proposition 2 yields that we can replace u with another sequence of the same limiting energy which is affine near the upper boundary. The construction near the lower boundary is analogous.

3. Segment rigidity.

The principal idea in the proof of Proposition 2 is to construct a grid such that the function u is approximately an isometry on the vertices. Once this is shown, the linear interpolate can be used to construct another function $\tilde{u}$ which is affine at the boundary and which has the same limiting energy as u. The construction of the grid is done by constructing first a reference grid and then choosing vertices in the neighborhoods of the reference vertices in order to have rigid edges (or segments). This local construction step is made possible by Proposition 1 below.

For notational simplicity we formulate the statement only for the case $A = \mathrm{Id}$, the general form can be obtained with a change of variables. The geometry is illustrated in Figure 1.

Proposition 1. *We study the following geometry: With $r > 0$ and $\alpha \in (0, 1/8)$ we consider $x_0, y_0 \in \Omega \subset \mathbf{R}^2$ such that $B((x_0 + y_0)/2, r) \subset \Omega$ and $r < |x_0 - y_0| < 2r - 4\alpha r$. Let furthermore a matrix $B \in \mathbf{R}^{2\times2} \setminus SO(2)$ be given and a function $\phi : \mathbf{R}^{2\times2} \to \bar{\mathbf{R}}$ with $\phi(F) \geq \bar{c}\,\mathrm{dist}^2(F, SO(2))$.*

Then for every $\theta > 0$ there are η and c (depending only on $\alpha, B, \bar{c}$ and θ) such that the following holds. For every $u \in C^1(\Omega, \mathbf{R}^2)$ with $\det \nabla u > 0$ *and*

$$\frac{1}{r^2}\int_\Omega \phi(\nabla u)dx + \frac{1}{r}\int_\Omega |\nabla\phi(\nabla u)| \leq \eta \tag{12}$$

there is a subset of $(x, y) \in B(x_0, \alpha r) \times B(y_0, \alpha r)$ with measure at least $1 - \theta$ the total such that

$$1 - c\varepsilon \leq \frac{|u(x) - u(y)|}{|x - y|} \leq 1 + c\varepsilon \tag{13}$$

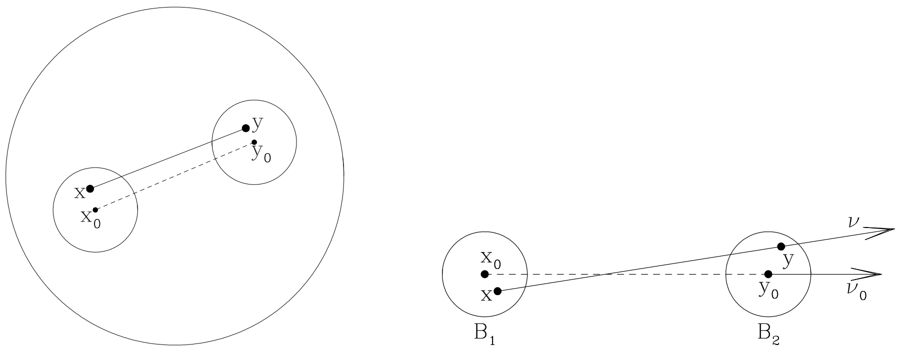

Figure 1: Sketch of the geometry in Proposition 1. We find balls around x_0 and y_0, such that many x and y in these balls have the desired properties. The vectors $y - x$ all have approximately the same direction. Figure from [5].

where

$$\varepsilon = \frac{1}{r^2} \int_{B_r} \mathrm{dist}(\nabla u, SO(2) \cup SO(2)B) dx \,. \tag{14}$$

Proof. Loosely speaking, inequality (12) provides that u is mostly in the A-phase, i.e. on the larger part of the domain ∇u is closer to $SO(2)A$ than to $SO(2)B$. Therefore, on most of Ω, ε controls indeed the distance of ∇u from $SO(2)A = SO(2)$, and we can expect that lengths are indeed shortened. Performing the same argument on the inverse of u (which exists, since u has positive determinant), lengths are approximately preserved.

We divide the proof in several steps. First we show that restricting the function to a slightly smaller ball it is invertible. Then we we define a 'bad' set Ω_{LG} of points where ∇u is far from a rotation, and show that Ω_{LG} is small and has small perimeter. By continuity, the gradient is still good on the boundary of the 'bad' set, hence the image of its boundary is also small. This shows that for a large fraction of the possible pairs (x, y), both segments $[x, y]$ and $[u(x), u(y)]$ do not intersect the 'bad' set (or its image). A straightforward line integration concludes the proof. Without loss of generality we can assume $r = 1$ (by scaling), $x_0 + y_0 = 0$ (by a translation), and $\Omega = B(0, 1)$ (restricting u).

Step 0. On a smaller domain, u is invertible. In order to derive the upper bound in (13) we connect the points x and y with the segment $[x, y]$ and calculate the length of the curve $u([x, y])$. The lower bound in (13) requires that we consider the segment $[u(x), u(y)]$ in the image and the curve in the pre-image $u^{-1}([u(x), u(y)])$. In this step of the proof we make sure that we can indeed invert u on such segments.

The quantitative rigidity estimate of Friesecke, James and Müller [8] asserts that for each function u there is a rotation $Q \in SO(2)$, such that the L^2 distance of ∇u to Q is controlled by the L^2 distance of ∇u to $SO(2)$. We combine this result with our assumptions on ϕ and find

$$\int_\Omega |\nabla u - Q|^2 dx \leq c \int_\Omega \mathrm{dist}^2(\nabla u, SO(2)) dx \leq c \int_\Omega \phi(\nabla u) dx \leq c\eta \,.$$

By the Poincaré inequality, u is close to an isometry $I(x) = Qx + b$,

$$\int_\Omega |\nabla u - \nabla I|^2 + |u - I|^2 \, dx \leq c\eta \,. \tag{15}$$

In particular, we find a radius $r' \in (1-\alpha/2, 1)$ such that

$$\int_{\partial B'} |\nabla u - \nabla I|^2 + |u - I|^2 \, d\mathcal{H}^1 \le c\eta \, ,$$

where $B' = B(0, r')$. By the embedding of $W^{1,2}(\partial B')$ in $C^0(\partial B')$, we have

$$|u(x) - I(x)| < c\eta^{1/2} \qquad \text{for all } x \in \partial B' \, . \tag{16}$$

Therefore, $u(\partial B')$ is uniformly close to a circle. We conclude that $u(B')$ contains a ball $B'' = B(I(0), r'')$ with $r'' \ge r' - c\eta^{1/2}$. Furthermore, the winding number of $u : \partial B' \to \mathbf{R}^2$ is one for each point in B''. From $\det \nabla u > 0$ we conclude that each point in B'' has exactly one pre-image in B'. Decreasing r' further by at most $\alpha/2$ we find a new ball $\tilde{B}'$ whose image is contained in B'', then u is invertible as a map

$$u : \tilde{B} \to u(\tilde{B}).$$

Repeating the same arguments, for small $\eta > 0$, the images $u(B(x_0, \alpha r))$ and $u(B(x_0, \alpha r))$ are contained in $\tilde{B}$. Since $\tilde{B}$ is convex, we always find a unique pre-image for segments $[u(x), u(y)]$ with $x \in B(x_0, \alpha r)$ and $y \in B(y_0, \alpha r)$.

Step 1. Definition of the bad set Ω_{LG}. By the coarea formula for BV functions, from (12) we obtain

$$\eta \ge |\nabla \phi(\nabla u)|(\Omega) = \int_{\mathbf{R}} \mathrm{Per}_\Omega(\{x \in \Omega : \phi(\nabla u(x)) \ge t\}) dt \, .$$

Therefore for any fixed $c_1 > 0$ there is $c_2 \in (c_1/2, c_1)$ such that the set

$$\Omega_{LG} = \{x \in B' : \phi(\nabla u(x)) \ge c_2\}$$

satisfies the bound

$$\mathcal{H}^1 \left(\partial \Omega_{LG} \cap B'\right) = \mathrm{Per}_{B'}(\Omega_{LG}) \le \frac{1}{c_1} \eta \, .$$

We choose c_1 small enough so that $\phi(F) \le c_1$ implies the inequality $\mathrm{dist}(F, SO(2)) < \min(\mathrm{dist}(F, SO(2)B), 1/4)$. The estimate of the first integral in (12) assures additionally $|\Omega_{LG}| \le c\eta$.

The gradient ∇u is uniformly close to $SO(2)$ on $\Omega \setminus \Omega_{LG}$ by definition. By continuity of ∇u this holds also on the boundary. In particular, $u(\partial \Omega_{LG})$ is rectifiable and

$$\mathcal{H}^1 \left(u(\partial \Omega_{LG} \cap B')\right) \le c\eta \, .$$

By the isoperimetric inequality, $u(\Omega_{LG})$ has also small area, bounded again by $c\eta$.

Step 2. Choice of x and y. In the following we show that several properties are satisfied by a large fraction of the possible choices of x and y in the balls $B(x_0, \alpha)$ and $B(y_0, \alpha)$. We show that each of them holds outside of a small set, and since we consider less than 10 properties, all of them will hold outside of a single, small set, whose area tends to zero as $\eta \to 0$. Therefore for any θ we can choose η sufficiently small, so that the thesis holds for sufficiently many points.

First, since $|\Omega_{LG}| \le c\eta$ we have that most points do not belong to it. We now show that for most pairs (x,y)

$$\int_{[x,y]} \operatorname{dist}(\nabla u, SO(2)) d\mathcal{H}^1 \le c\varepsilon\,, \tag{17}$$

where c depends on α and θ. We first consider the integral of $\operatorname{dist}(\nabla u, K)$. Its integral on the entire domain is controlled by ε, hence – choosing a suitable c – on a large fraction of the lines through the domain it is controlled by $c\varepsilon$ (to formalize this step it helps to prove it first by prescribing the direction of the segment). At the same time, since $\partial\Omega_{LG}$ is small, few lines can intersect it. But since x and y are outside of Ω_{LG}, for most choices the entire segment $[x,y]$ is outside Ω_{LG}. Since outside Ω_{LG} we have $\operatorname{dist}(\nabla u, SO(2)) = \operatorname{dist}(\nabla u, K)$, this concludes the proof of (17).

Now we consider analogously the segments $[u(x), u(y)]$ in the image. Since $u(\partial\Omega_{LG} \cap B')$ has small length, most segments $[u_x, u_y]$ do not intersect $u(\partial\Omega_{LG} \cap B')$ (we denote by u_x a generic point in $u(B(x_0,\alpha))$, and analogously u_y). Outside Ω_{LG} the determinant of ∇u is bounded from above and away from zero, hence the measure of exceptional pairs (u_x, u_y) in the image can be compared with the measure of exceptional pairs (x,y) with $u(x) = u_x$ and $u(y) = u_y$. We find that for most pairs (x,y) the segment $S = [u(x), u(y)]$ does not hit $u(\Omega_{LG})$. By Step 0, given S, we find a curve $\gamma_{xy} : [0,1] \to B \setminus \Omega_{LG}$ such that $u \circ \gamma_{xy}$ is a C^1 monotonic parametrization of S.

It remains to show that for most (x,y) the curve γ_{xy} carries energy of order ε. We define $g : \mathbf{R}^2 \to \mathbf{R}$ by

$$g(z) = \operatorname{dist}(\nabla u, K)(u^{-1}(z)) = f(u^{-1}(z))\,,$$

and find many segments $[u_x, u_y]$ such that integrals of g over $[u_x, u_y]$ are of order ε. Therefore, as above, also for most pairs (x,y), integrals of g over $[u(x), u(y)]$ are of order ε (we use that away from Ω_{LG} the Jacobian determinant is close to one). We conclude that for most pairs (x,y)

$$\int_{\gamma_{xy}} f\, d\mathcal{H}^1 \le c \int_{[u(x),u(y)]} g\, d\mathcal{H}^1 \le c\varepsilon\,. \tag{18}$$

Step 3. Length estimates. We have shown that

$$\int_{[x,y]} \operatorname{dist}(\nabla u, SO(2))\, d\mathcal{H}^1 \le c\varepsilon \quad \text{and} \quad \int_{\gamma_{xy}} \operatorname{dist}(\nabla u, SO(2))\, d\mathcal{H}^1 \le c\varepsilon\,,$$

where γ_{xy} is a C^1 curve joining x and y, with $u \circ \gamma_{xy}$ being a monotonic parametrization of the segment $[u(x), u(y)]$. The first condition implies

$$\begin{aligned} |u(x) - u(y)| &\le \int_{[x,y]} |\nabla_\tau u|\; d\mathcal{H}^1 \\ &\le |x-y| + \int_{[x,y]} \operatorname{dist}(\nabla u, SO(2)) d\mathcal{H}^1 \\ &\le |x-y| + c\varepsilon\,, \end{aligned}$$

where ∇_τ denotes the tangential derivative. Since u is a one-to-one map of the curve γ_{xy} onto the segment $[u(x), u(y)]$,

$$|u(x) - u(y)| \;=\; \int_{\gamma_{xy}} |\nabla_\tau u|\, d\mathcal{H}^1$$

$$\begin{aligned} &\geq \mathcal{H}^1(\gamma_{xy}) - \int_{\gamma_{xy}} \operatorname{dist}(\nabla u, SO(2)) d\mathcal{H}^1 \\ &\geq |x-y| - c\varepsilon\,. \end{aligned}$$

This concludes the proof of Proposition 1. q.e.d.

4. Construction of a recovery sequence.

4.1 Construction of rigid grids.

Proposition 2. *Let* $\Omega = (-d,d) \times (-l,l)$ *be a rectangle in* $\mathbf{R}^2$*. Then there are constants* $c, \eta, \varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$ *and every function* $u : \Omega \to \mathbf{R}^2$ *with*

$$I_\varepsilon[u,\Omega] = \int_\Omega \frac{1}{\varepsilon} W(\nabla u) + \varepsilon |\nabla^2 u|^2 \, dx \leq \eta \tag{19}$$

there exists a function $\tilde{u} : \tilde{\Omega} \to \mathbf{R}^2$ *on* $\tilde{\Omega} = (-d/4, d/4) \times (-l,l)$ *with*

$$I_\varepsilon[\tilde{u}, \tilde{\Omega}] \leq c\eta \tag{20}$$

which is affine in $(-d/4, d/4) \times (l/2, l)$*.*

The proof is based on the segment rigidity of Proposition 1. We first observe that $\nabla u \in W^{1,2}$ together with $\det \nabla u > 0$ a.e. implies $\nabla u \in C^0$ by a result of Vodop'yanov and Gol'dshtein, [10], see also Šverák [11], hence the smoothness assumption in Proposition 1 is satisfied. Since we are dealing with an interior estimate, by restricting to a slightly smaller domain we can also assume that $|\nabla u| \leq M$ and $\det \nabla u > 1/M$ for some $M > 0$.

The function ϕ used in Proposition 1 is constructed from the geodesic distance $d_W(F,G)$ induced on $\mathbf{R}^{2\times 2}$ by the potential W,

$$\begin{aligned} d_W(F,G) \;=\; \inf \Big\{ \int_0^1 \sqrt{W(g(s))} |g'(s)| ds : g \in C^0([0,1], \mathbf{R}^{2\times 2}), \\ g(0) = F, g(1) = G, g \text{ piecewise } C^1 \Big\}\,. \end{aligned} \tag{21}$$

We observe that $d_W(F,A) = 0$ iff $F = QA$ for some $Q \in SO(2)$, and the same for B. If $AB^{-1} \notin SO(2)$, since W is positive away from K we get $d_W(B,A) > 0$. Further, the function $d_W(\cdot, A)$ is C^1 smooth and globally Lipschitz on the set of matrices $\{F : |F| \leq M$ and $\det F > 1/M\}$ in which ∇u takes values, and its derivative satisfies $|\nabla d_W(\cdot, A)| \leq \sqrt{W(\cdot)}$. Therefore

$$\int_\Omega |\nabla d_W(\nabla u(x), A)| \leq \int_\Omega \sqrt{W(\nabla u(x))} \left|\nabla^2 u(x)\right| dx \leq \frac{1}{2} I_\varepsilon[u,\Omega]\,.$$

This implies that $d_W(\nabla u_i(x), A)$ is uniformly bounded in $W^{1,1}$. We consider now the case that A is the majority phase of u. We set

$$\phi(x) := d_W(\nabla u(x), A),$$

and infer $\|\phi\|_{W^{1,1}} \leq c\eta$. By a change of variables we can assume $A = \mathrm{Id}$. As in [4], Lemma 4.5, we find ζ_0 in $(0, l/2)$ with

$$\frac{1}{\delta} \int_{(-d,d)\times(\zeta_0 - \delta, \zeta_0)} \phi + |\nabla\phi| \leq c\eta \tag{22}$$

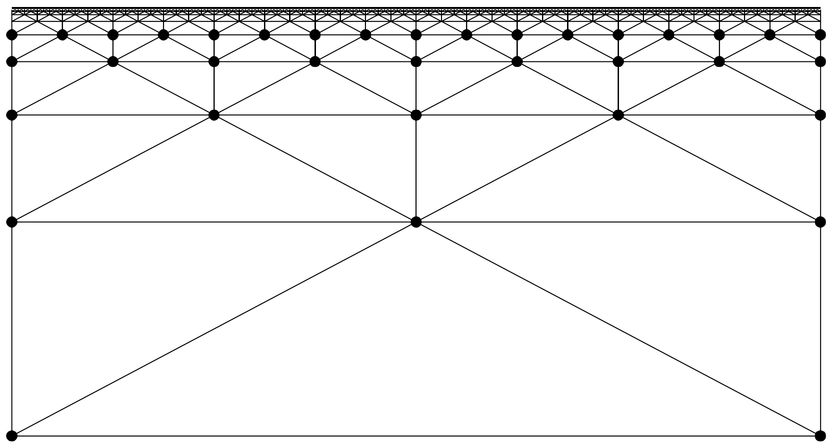

Figure 2: The reference grid, for a rectangle with aspect ratio 3.

for all $\delta \in (0, l/2)$. This estimate will enable us to apply Proposition 1 to u on small subsets. We note here that $\phi \geq \bar{c}\mathrm{dist}^2(., K)$ is satisfied for some positive $\bar{c}$, since W has quadratic growth by (8).

Construction of the reference grid. We construct a triangular grid as indicated in Figure 2. Precisely, we first map affinely the domain into the rectangle $(0,1) \times (0, \sqrt{3})$, then construct on this rectangle a triangular grid which refines towards the top face and only contains triangles which are either equilateral or have angles of 30, 60 and 90 degrees, then transform back. The resulting grid refines towards the line $\Gamma_0 = (-d, d) \times \{\zeta_0\}$ and contains triangles whose angles are uniformly bounded away from zero (with a bound depending on the aspect ratio l/d). The vertical spacing between the grid lines is given, at stage k, by $h_k = c2^{-k}$, where c is a (universal) geometric constant, and all lengths at refinement stage k are uniformly controlled from above and below by h_k (with constants depending on l/d).

For later use we now show that (22) implies a local control of ϕ in L^1 with the optimal scaling. We consider an interval $Y_k = (\zeta_0 - h_k, \zeta_0)$ and rewrite (22) as

$$\int_{Y_k} e(\zeta)\, d\zeta \leq c\eta h_k\,, \quad \text{where} \quad e(\zeta) = \int_{-d}^{d} \phi(\xi, \zeta) + |\nabla\phi(\xi, \zeta)|\, d\xi\,.$$

Hence there is $\zeta_k \in Y_k$ such that $e(\zeta_k) \leq c\eta$. The one-dimensional embedding $W^{1,1} \subset L^\infty$ yields that $\phi(\cdot, \zeta_k)$ is bounded by $c'\eta$ pointwise on $(-d, d)$. The Poincaré inequality for the square $K = (\xi, \xi + h_k) \times (\zeta_0 - h_k, \zeta_0) = (\xi, \xi + h_k) \times Y_k$ yields, for arbitrary $\xi \in (-d, d - h_k)$,

$$\int_K \phi\, dx \leq h_k^2 \|\phi(\cdot, \zeta_k)\|_{L^\infty(-d,d)} + h_k \int_K |\nabla\phi|\, dx \leq c\eta h_k^2. \tag{23}$$

Construction of the perturbed grid. Our aim is to apply Proposition 1 to pairs of neighboring vertices with $\alpha = 1/10$. Since the assertion of Proposition 2 is a statement for a subdomain, we can restrict the grid such that for each vertex v_m of level k, the ball $B_m = B(v_m, \alpha l_k)$ is contained in Ω. Let $n = (m, m')$ denote one pair of neighboring vertices at level k. Then by (23) we can apply Proposition 1 to it, provided that η is chosen appropriately. Hence there are many pairs $(w_m, w_{m'}) \in B_m \times B_{m'}$ such that the segment $[w_m, w_{m'}]$ is rigid.

We now choose inductively one point w_m in each B_m. We start by saying that all points of all balls are *possible choices in step* 0. At step m we set w_m as one of the *possible choices in step* m in the ball B_m with the additional condition that w_m forms a rigid pair with many points of all neighboring balls $B_{m'}$ with $m' > m$. This is possible provided that in B_m there are many possible choices in step m, since there is only a finite number of neighbors. The set of *possible choices in step* $m+1$ then consists of the possible choices in step m without those $w \in B_{m'}$ such that $m' > m$ is a neighbor of m, but (w, w_m) is not a rigid pair. Since each point has at most seven neighbors, we reduce the possible choices of each ball B_m finitely many times by a small set, whence at all steps there remain many possible choices in each ball where the choice has not been made yet.

4.2 The matching function in the impenetrable case.

With inequalities (22) and (23) we have verified the smallness assumptions of Proposition 1 for every edge $e_n = [x, y]$ of level k. Concerning the error (defined in (14)), we use a rectangle $R_n = (\xi, \xi + h_{k-1}) \times Y_{k-1}$ of level k which contains e_n and obtain

$$\left| \frac{|u(x) - u(y)|}{|x - y|} - 1 \right| \leq c \frac{1}{h_k^2} \|\mathrm{dist}(\nabla u, K)\|_{L^1(R_n)} \leq c \frac{1}{h_k} \|\mathrm{dist}(\nabla u, K)\|_{L^2(R_n)}. \tag{24}$$

As a first step in the construction of $\tilde{u}$ we define a linear interpolate v by setting $v(x) = u(x)$ for each vertex x of the perturbed grid, and v affine on each triangle T_m of the perturbed grid. By (24), u is approximately length-preserving on edges, therefore v is close to a rigid motion on each triangle. In terms of inequalities we find that for each triangle T_m there exists a rotation $Q_m \in SO(2)$ such that

$$|\nabla v(T_m) - Q_m| \leq c \frac{1}{h_k} \sum_{i=1}^{3} \|\mathrm{dist}(\nabla u, K)\|_{L^2(K_{n_i})}, \tag{25}$$

where $\nabla v(T_m)$ is the value of ∇v on T_m, and the K_{n_i} are rectangles of level k as above which cover the three edges e_{n_i} of T_m. We note that v has some of the properties that we require from $\tilde{u}$ in Proposition 1. It is identical to u on the line $(-d/4, d/4) \times \{\zeta_0\}$, it is affine on the line $(-d/4, d/4) \times \{\zeta_0 - h_1\}$, and the bulk part of the energy is comparable to that of the function u. Yet we can not use v as the function $\tilde{u}$, since v is not a $W^{2,2}$-function and has therefore an infinite surface energy.

Definition of the $W^{2,2}$-function $\tilde{v}$. We will construct $\tilde{v}$ as a piecewise polynomial function interpolating values of v and of ∇v. In order to define $\tilde{v}$ we use interpolation defined by the Argyris triangle, which is well-known in numerical analysis (other names are Bell's triangle and 21-degree of freedom triangle, see [1]). We will now briefly recall the construction of this interpolation on triangles.

Given a triangle T we denote the corners by (p_i), $i = 1, ..., 3$, and the edges by e_{ij}, $i \neq j$, $i, j \in \{1, ..., 3\}$. To each edge e_{ij} we associate the mid-point $p_{ij} = (p_i + p_j)/2$. We can now define a polynomial interpolation $F : T \to \mathbf{R}$ of given point-values as follows. Prescribing F, ∇F, and $\nabla^2 F$ in the corners (p_i) (18 degrees of freedom), and additionally the normal derivatives $\nu \cdot \nabla F$ of F in the mid-points p_{ij} (3 degrees of freedom) we find a polynomial F of degree 5 with these point-values. Indeed, there holds:

Lemma 1. The Argyris interpolation has the following properties.

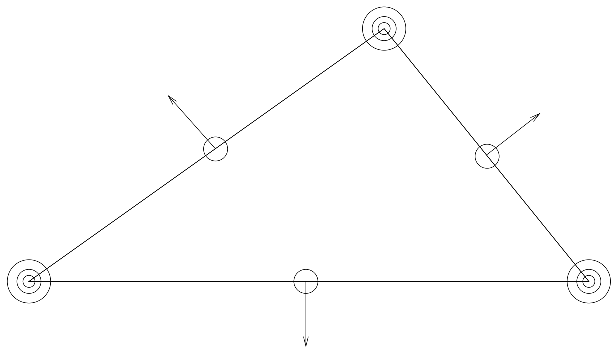

Figure 3: The Argyris triangle

a) Given a triangle T and 21 point-values as above, there exists a unique polynomial F of degree 5 having the prescribed point-values.

b) For point-values $\xi_i \in \mathbf{R}$, $i = 1, ..., 3$, and $\zeta_i \in \mathbf{R}^2$, $i = 1, ..., 6$, let $F : T \to \mathbf{R}$ be the polynomial of degree 5 satisfying

$$F(p_i) = \xi_i, \quad \nabla F(p_i) = \zeta_i, \quad \nabla^2 F(p_i) = 0 \in \mathbf{R}^{2\times 2}, \quad i = 1, ..., 3.$$
$$\nu \cdot \nabla F(p_{23}) = \nu \cdot \zeta_4, \quad \nu \cdot \nabla F(p_{31}) = \nu \cdot \zeta_5, \quad \nu \cdot \nabla F(p_{12}) = \nu \cdot \zeta_6.$$

The map $\mathbf{R}^{15} \ni (\xi, \zeta) \mapsto F \in C^2(T)$ on a triangle with diameter h has the following property. We set $\bar{F}$ to be the affine interpolation of the values ξ_i and $\zeta_0 := \nabla \bar{F}$. There holds

$$\|\nabla^2 F\|_{C^0} \le c\frac{1}{h} \sup\{|\zeta_k - \zeta_l| : k, l = 0, ..., 6\}. \tag{26}$$

The constant c depends only on bounds for the aspect ratio of T.

c) Let (T_m) be a triangulation of $D = \bigcup_m T_m$ and let point-values be defined on vertices and mid-points of edges. Then the piece-wise polynomial Argyris interpolation yields a C^1-function on D.

Proof. In order to prove a) it suffices to show that the linear space of polynomials of order 5 in two variables has exactly dimension 21, and that the 21 linear conditions imposed by the point values are linearly independent. Analogously, c) follows from the same argument done for the restriction of a generic polynomial to the side which is common to two neighboring triangles. For details we refer to [1], Theorem 2.2.11 and Theorem 2.2.13. For the proof of b) we first note that $(\xi_1, ..., \xi_3, \zeta_1, ..., \zeta_6)$ and $(\xi_1, \zeta_0, ..., \zeta_6)$ are equivalent sets of parameters. In the second set of parameters, the left hand side of (26) is independent of ξ_1, and independent upon adding a constant to all ζ_j. Hence $\nabla^2 F$ depends only on the indicated differences $\zeta_k - \zeta_l$. Since both sides scale as h^{-1} under dilations of the triangle by a factor h, it suffices to consider triangles of unit diameter. After this reduction of the proof we can now define $c > 0$ to be the supremum of $\|\nabla^2 F\|_{C^0}$ over all ζ with $|\zeta_k - \zeta_l| \le 1$ and over all triangles with unit diameter and uniformly bounded aspect ratio. By continuity of $\|\nabla^2 F\|_{C^0}$ in this finite number of parameters, we find a finite supremum c. q.e.d.

We will now define $\tilde v$ on the rigid (the perturbed) grid. We will use the function ∇v that was already constructed as the piecewise linear interpolation. Since ∇v has jumps along edges, the function ∇v does not directly provide us with values in vertices p and in mid-points q of edges. But we can associate to each p one of the (at most 7) triangles that have p as a corner, and we can associate to the mid-point q of each edge e one of the 2 triangles that have e as an edge. We can now define point-values of a function $\tilde v$ by setting $\tilde v(p) = v(p)$, $\nabla \tilde v(p) = \nabla v(T)$ in vertices, and $\nabla \tilde v(q) = \nabla v(T)$ in mid-points, where in both definitions ∇v is evaluated in the associated triangle T to find $\nabla v(T)$. We define the function $\tilde v$ on each triangle as the interpolation of these values as in part b) of Lemma 1 (in particular, we consider vanishing second gradients in vertices and use only the normal derivative in mid-points). Part a) of Lemma 1 yields the existence of $\tilde v$ and part c) the property $\tilde v \in C^1$, hence $\tilde v \in W^{2,2}((-d/4, d/4) \times \{\zeta_0 - h_1\})$.

Calculation of energies of $\tilde v$.

1) The bulk energy. The estimate for the energy $W(\nabla \tilde v)$ is obtained from the growth condition from (9) in two steps. We first show that ∇v is uniformly close to $SO(2)$, which permits us to use (9). This first step is done starting from the local estimate (23) on ϕ. In a second step, we show that the distance of ∇v from $SO(2)$ is, in an L^2 sense, small compared to ε. This will be based on using the smallness of the original bulk energy $W(\nabla u)$, and will lead to the desired estimate.

We start by combining the ∇v-estimate of (25) with the ϕ-estimate of (23), to obtain the pointwise inequality

$$|\nabla v(T_m) - Q_m| \le c \frac{1}{h_k} \left(\int_S \phi \, dx \right)^{1/2} \le c\sqrt{\eta} \tag{27}$$

for an appropriate choice of S. We remark that this is a weaker estimate (by a factor $\varepsilon^{-1/2}$) in a stronger norm (L^∞ instead of L^2) than the one that can be obtained combining (25) with the assumption on $W(\nabla u)$.

Analogously, we can control pointwise the difference between the rotations Q_m and $Q_{m'}$ of two neighboring triangles T_m and $T_{m'}$ with common edge $e_n = [x, y]$. Since $v|_{T_m}$ and $v|_{T_{m'}}$ coincide on the direction $\bar e_n$ of the segment e_n, $\nabla v_m \cdot \bar e_n = \nabla v_{m'} \cdot \bar e_n$, also the rotations on both sides are comparable up to an error as in (25).

$$|Q_m - Q_{m'}| \le c \frac{1}{h_k} \sum_{i=1}^{5} \|\mathrm{dist}(\nabla u, K)\|_{L^2(K_{n_i})}, \tag{28}$$

where now n_i are related to the five distinct edges of the two triangles T_m and $T_{m'}$. Note that the estimate can be iterated to triangles T_m and $T_{m'}$ which have only one vertex in common. In the estimate (28) we then sum over the seven distinct edges of three triangles. We conclude from (27) and Part b) of Lemma 1 that also the C^1-function $\nabla \tilde v$ is pointwise close to $SO(2)$. By (9), we can therefore always compare W with $\mathrm{dist}(., SO(2))^2$ and find

$$\begin{aligned} \int \frac{1}{\varepsilon} W(\nabla \tilde v) dx &\le c \frac{1}{\varepsilon} \int |\mathrm{dist}(\nabla \tilde v, SO(2))|^2 dx \\ &\le c \frac{1}{\varepsilon} \int |\mathrm{dist}(\nabla v, SO(2))|^2 dx + c \frac{1}{\varepsilon} \int |\mathrm{dist}(\nabla u, K)|^2 dx \le c\eta. \end{aligned} \tag{29}$$

2) The surface energy. We first draw conclusions from the small surface energy of u. Each triangle T_m of level k is contained in a rectangle K_m of level k. On K_m we can approximate u by the affine function F_m defined by the averages of u and ∇u. The Sobolev embedding yields

$$\|u - F_m\|^2_{L^\infty(K_m)} + \int_{T_m} |\nabla u - \nabla F_m|^2 dx \leq c h_k^2 \int_{K_m} |\nabla^2 u|^2 dx. \tag{30}$$

Since u is continuous, this is a pointwise estimate for $u - F_m$ in the vertices of T_m. But v is defined on T_m by the values of u in the vertices, therefore

$$|\nabla v(T_m) - \nabla F_m| \leq \|u - F_m\|_{L^\infty(K_m)} \frac{1}{h_k} \leq c \left(\int_{K_m} |\nabla^2 u|^2 dx \right)^{1/2}. \tag{31}$$

The two affine maps F_m and $F_{m'}$ on neighboring triangles can be compared again by second derivatives of u and we find

$$|\nabla v(T_m) - \nabla v(T_{m'})| \leq c \left(\int_{K_m \cup K_{m'}} |\nabla^2 u|^2 dx \right)^{1/2}. \tag{32}$$

We can now use Lemma 1b) to calculate in a fixed triangle T_{m_0} of level k with neighbors T_{m_i}, $i \geq 1$,

$$\int_{T_{m_0}} \varepsilon |\nabla^2 \tilde{v}|^2 dx \quad \leq \quad \varepsilon h_k^2 \, c^2 \sup_{i \geq 1} \frac{|\nabla v(T_{m_0}) - \nabla v(T_{m_i})|^2}{h_k^2} \leq c\varepsilon \sum_{i=1} \int_{K_{m_i}} |\nabla^2 u|^2 dx.$$

Summing now over all triangles m_0 we find as estimate for the surface energy

$$\int \varepsilon |\nabla^2 \tilde{v}|^2 dx \leq c \int \varepsilon |\nabla^2 u|^2 dx \leq c\eta. \tag{33}$$

This is the desired estimate for $\tilde{v}$.

3) The function $\tilde{u}$.

The function $\tilde{v}$ is of class $W^{2,2}$ and has the correct scaling of the energy. But, still, we have not found the desired function $\tilde{u}$, since $\tilde{v}$ is not affine on the segment $(-d/4, d/4) \times \{\zeta_0 - h_1\}$ contained in the triangle T_o. We define $\tilde{u}$ as

$$\begin{aligned} \tilde{u} &= \tilde{v} \quad \text{on } \Omega \setminus T_o, \\ \tilde{u} &= v \quad \text{near } (-d/4, d/4) \times \{\zeta_0 - h_1\}, \end{aligned}$$

and interpolate between these values. The interpolation increases the energy at most by $c\eta$ by calculations as in step 2) above. This concludes the proof of Proposition 2. q.e.d.

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