Rigidity and Gamma convergence for solid–solid phase transitions with $SO(2)$–invariance

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The singularly perturbed two-well problem in the theory of solid-solid phase transitions takes the form

$$I_\varepsilon[u] = \int_\Omega \frac{1}{\varepsilon} W(\nabla u) + \varepsilon |\nabla^2 u|^2,$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is the deformation, and W vanishes for all matrices in $K = SO(n)A \cup SO(n)B$. We focus on the case $n = 2$ and derive, by means of Gamma convergence, a sharp-interface limit for I_ε. The proof is based on a rigidity estimate for low-energy functions. Our rigidity argument also gives an optimal two-well Liouville estimate: if ∇u has a small BV norm (compared to the diameter of the domain), then, in the L^1 sense, either the distance of ∇u from $SO(2)A$ or the one from $SO(2)B$ is controlled by the distance of ∇u from K. This implies that the oscillation of ∇u in weak-L^1 is controlled by the L^1 norm of the distance of ∇u to K.

1 Introduction

In the gradient theory of solid-solid phase transitions one considers energies of the form

$$I_\varepsilon[u,\Omega] = \int_\Omega \frac{1}{\varepsilon} W(\nabla u) + \varepsilon |\nabla^2 u|^2 dx \tag{1.1}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ stands for the deformation, and the free energy density $W(F)$ is nonnegative and satisfies

$$W(F) = 0 \quad \text{iff} \quad F \in K = SO(n)A \cup SO(n)B\,.$$

Here A and B are two fixed, rank-one connected invertible matrices in $\mathbb{R}^{n\times n}$ and $SO(n)$ is the set of rotations in $\mathbb{R}^n$ [4, 10, 28, 32, 6]. In this paper we restrict to two spatial dimensions and focus on the frame–indifferent case, where $W(QF) = W(F)$ for all $Q \in SO(2)$ and all $F \in \mathbb{R}^{2\times 2}$.

In the limit $\varepsilon \to 0$ one expects that ∇u takes values only in K, and that the energy is proportional to the length of the interfaces between the region where $\nabla u \in SO(2)A$ and the one where $\nabla u \in SO(2)B$. The notion of convergence appropriate for this kind of limiting problems is Gamma convergence, as developed by De Giorgi and his school in the 70s ([15]; see also [14, 8]). Whereas the functionals which arise in fluid-fluid phase transitions, of the form

$$J_\varepsilon[v,\Omega] = \int_\Omega \frac{1}{\varepsilon} W(v) + \varepsilon |\nabla v|^2\, dx, \tag{1.2}$$

have been thoroughly studied [31, 19, 29, 7, 34, 5], the gradient structure characteristic of problems in elasticity has proven more difficult to understand. A Gamma convergence result for the functional I_ε was obtained, under the assumption that W vanishes only on two matrices (i.e. neglecting rotational invariance), in [12]. Inclusion of rotational invariance within a linearized setting was achieved, in two dimensions, in [13]. Here we present the first result which fully includes rotational invariance in a nonlinear setting (Theorem 3.1).

The proof of Gamma convergence for the $SO(2)$-invariant problem presents two new difficulties. The first is in deriving compactness, where the control of oscillations between the wells (obtained via the classical Modica-Mortola argument) has to be combined with a rigidity argument to control oscillations within the wells. The structure of limiting deformations is then determined through the characterization of functions u with $\nabla u \in BV(\Omega, K)$ obtained by Dolzmann and Müller [16] (see Proposition 3.2). The second difficulty lies in deriving the upper bound, which requires the construction of sequences of optimal energy approximating a limit which has multiple flat interfaces. This is done combining a choice of an optimal sequence for each interface with an interpolation, which is made possible by an optimal estimate for the deviation of low-energy deformations from affine maps in $H^{1/2}$ on lines (Proposition 3.4). This estimate permits to extend the function with small H^1 norm (Proposition 3.5). The key ingredient in the proof of the $H^{1/2}$ rigidity is a two-well rigidity result on segments (Proposition 2.2).

The rigidity of segments permits also to obtain an optimal estimate on the amount of minority phase present away from interfaces. We show (Theorem 2.1) that if $|\nabla^2 u|(\Omega)$ is small compared to $\mathrm{dist}(\Omega', \partial\Omega)$, where Ω' is a connected subset of Ω, then

$$\min_{J\in\{A,B\}} \|\mathrm{dist}(\nabla u, SO(2)J)\|_{L^1(\Omega')} \leq c\|\mathrm{dist}(\nabla u, K)\|_{L^1(\Omega)} \,. \tag{1.3}$$

A variation of the one-well rigidity estimate by Friesecke, James, and Müller [21] shows that (1.3) implies an equivalent control on the weak-L^1 norm of $\nabla u - F$, for some $F \in K$ (Proposition 2.6). This improves a previous result by Lorent [30], who under the additional assumptions that u is bilipschitz and $\det A = \det B$ obtained that the minimum of $\|\mathrm{dist}(\nabla u, F)\|_{L^1(\Omega')}$ over $F \in K$ is controlled by a power of $\|\mathrm{dist}(\nabla u, K)\|_{L^1(\Omega)}$. In the limiting case where u satisfies $\nabla u \in K$ a.e. with $\nabla^2 u \in BV$, Dolzmann and Müller [16] have proven that u has locally the structure of a laminate. Our result constitutes an optimal quantitative version of this rigidity result, and we expect that it will be useful in the derivation of lower bounds for singularly perturbed two-well problems.

The geometric arguments used here to obtain two-well rigidity are extensions of previous ideas used by John [24] and Kohn [27] to obtain single-well rigidity results, of those used by Dolzmann and Müller [16] for the rigid case $\nabla u \in BV(\Omega, K)$, and of those used in our previous work [13] on the geometrically linear problem. Our approach is rather different from the PDE approach which recently permitted to Friesecke, James, and Müller [21] to obtain the quantitative one-well rigidity estimate.

In Section 2 we present the geometric rigidity arguments, and in Section 3 the Gamma convergence, including the $H^{1/2}$ rigidity. Section 3 depends only on the segment-rigidity presented in Subsection 2.2, not on the rest of Section 2.

2 Two-well rigidity in L^1 and segment rigidity

2.1 Introduction

A function $u : \Omega \to \mathbb{R}^2$ whose gradient takes values in $SO(2)$ is affine (on connected sets), as was shown by Liouville. If ∇u is close to $SO(2)$ in an L^p sense, then u is approximately affine in $W^{1,p}$, as was recently proven by Friesecke, James, and Müller [21] (see Section 2.4 for details). We are here interested in analogous two-well results, i.e. on functions $u : \Omega \to \mathbb{R}^2$ whose gradient is close to $K = SO(2)A \cup SO(2)B$.

If the two wells $SO(2)A$ and $SO(2)B$ are rank-one connected, in the sense that there are $Q, R \in SO(2)$ such that $\mathrm{rank}(QA - RB) = 1$, then there are piecewise affine functions with $\nabla u \in K$ which are not globally affine. For example, if $QA - B = a \otimes \nu$, let $\chi : \mathbb{R} \to \mathbb{R}$ be any Lipschitz function whose derivative takes only the values 0 and 1, and define

$$u(x) = Bx + a\chi(x \cdot \nu)\,.$$

Then $\nabla u \in K$ everywhere; ∇u can oscillate on an arbitrarily fine scale between QA and B, but all interfaces are straight (see Figure 2.1(a) for an example). In general, given two matrices A and B gradient fields taking values in K can jump only across planar interfaces, and the normal to the interface ν can take at most two values (we neglect here the degenerate case $AB^{-1} \in SO(2)$). This corresponds to the fact that $\mathrm{rank}(QA - B) = 1$ has at most two solutions for $Q \in SO(2)$. Each solution $QA - B = a \otimes \nu$ is called a rank-one connection between the two wells $SO(2)A$ and $SO(2)B$, see [4, 32]. The above construction shows that, if at least one rank-one connection is present, no global rigidity result can be expected without assuming some bound on the surface energy. A different picture arises for the case without rank-one connections (i.e. when $\mathrm{rank}(QA - B) = 1$ has no solutions for $Q \in SO(2)$), see Chaudhuri and Müller [9].

We consider the case that ∇u is close to K and the distributional second gradient $\nabla^2 u$ is small. The second assumption corresponds to interfaces being short, i.e. to the situation illustrated in Figure 2.1(b). It turns out that isolated regions of the minority phase cannot be realized without a significant cost in terms of the two-well energy $\int \mathrm{dist}(\nabla u, K)dx$. We will show that, upon taking a subdomain, the amount of minority phase can be estimated in terms of the two-well energy alone. This implies that one can replace the two-well energy with a single-well energy (in a subdomain). In applications it is useful to combine this with the corresponding single-well rigidity estimates discussed in Section 2.4 below.

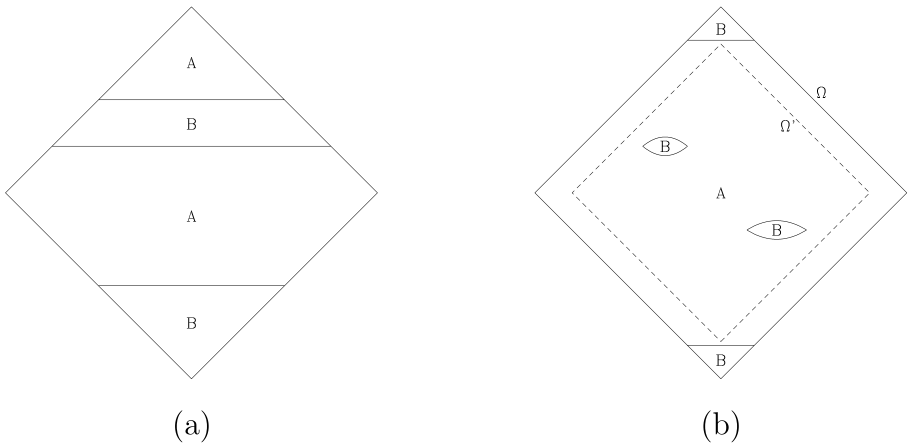

FIGURE 2.1: (a): There are nontrivial gradient fields ∇u which take values in K. They have a laminar structure, where interfaces are straight lines. (b): If the total length of the interfaces is small, and ∇u is close to K, one can have minority regions close to the boundary, and minority islands inside. The latter necessarily have curved boundaries, and are penalized also by the elastic energy. Minority regions near the boundary can have straight boundaries and zero elastic energy.

We first give a precise statement and then sketch the main ideas of the proof. The proof of Theorem 2.1 is then contained in Sections 2.2 and 2.3.

Theorem 2.1. *Let A and $B \in \mathbb{R}^{2\times 2}$ have positive determinant, $\Omega' \subset\subset \Omega \subset \mathbb{R}^2$ be two bounded Lipschitz domains, with Ω' connected. Then there are constants c_0, c_1, c_2, and η_0 such that for any $u \in W^{1,1}(\Omega, \mathbb{R}^2)$ with $\nabla u \in BV$ which satisfies*

$$\int_\Omega |\nabla^2 u| \le \eta_0(A,B)\mathrm{dist}(\Omega', \partial\Omega) \tag{2.1}$$

one has

$$\min_{J\in\{A,B\}} \int_{\Omega'} \mathrm{dist}(\nabla u, SO(2)J)dx \le c_0 \int_\Omega \mathrm{dist}(\nabla u, SO(2)\{A,B\})dx\,, \tag{2.2}$$

where $c_0 = c_0(\Omega, \Omega', A, B)$. Further, if

$$\int_\Omega \mathrm{dist}(\nabla u, SO(2)\{A,B\})dx \le c_2(A,B)\mathrm{dist}^2(\Omega', \partial\Omega) \tag{2.3}$$

then (2.2) holds with $c_0 = c_1(A,B)$, independent of the domains.

A first result in this direction was obtained by Lorent [30] for the case that $\det A = \det B$ and u is bilipschitz (i.e. Lipschitz with Lipschitz inverse).

Precisely, Lorent has shown that, on special domains, (2.1) implies

$$\min_{J\in K}\int_{\Omega'}|\nabla u - J|dx \le c\left[\int_{\Omega}\mathrm{dist}(\nabla u, SO(2)\{A,B\})dx\right]^{\gamma}.$$

Here $\gamma = 1/800$ and the constants depend on the Lipschitz norms of u and u^{-1}. For the case that A and B have no rank-one connections (incompatible wells) an L^2 version of this result was proven by Chaudhuri and Müller [9]; their result does not require assumption (2.1) on the higher derivative. In (2.1) and below se use the notation

$$\int_{\Omega}|\nabla\psi| = \sup\left\{\int_{\Omega}\psi\,\mathrm{div}\,\varphi\,dx:\;\;\varphi\in C^1_0(\Omega,\mathbb{R}^n)\,,\;\;|\varphi|\le 1\right\}.$$

In order to sketch the main ideas, we first prove the result for the case of an invertible C^1 function u, with the non-optimal scaling $\varepsilon^{1/2}$. Here and below we denote the right-hand side of (2.1) by η, and the integral on the right-hand side of (2.2) by ε. Further, we assume Ω to be the unit ball, and Ω' a sufficiently small ball centered in the origin. We write c for a generic numerical constant, which does not depend on the deformation u or on ε, but can change from line to line.

In the proof we choose points such that the segments joining them are 'as good as on average' (up to a constant) from several points of view. More precisely, if $\psi_i:\Omega\to\mathbb{R}$, $1\le i\le N$ are finitely many nonnegative functions which obey $\int\psi_i dx\le c_i|\Omega|$, and θ is any number in $(0,1)$, then there is $\omega\subset\Omega$ with $|\omega|\ge(1-\theta)|\Omega|$ such that

$$\text{for all } x\in\omega\quad\text{and all } i\qquad\text{one has } \psi_i(x)\le\frac{N}{\theta}\,c_i\,.$$

Indeed, the set $\Omega\setminus\omega$ where this property is violated coincides with the union of the sets $A_i=\{x:\psi_i(x)>Nc_i/\theta\}$, and each of them has size controlled by $\theta|\Omega|/N$. In the rest of this section we shall not state the value of θ precisely, but say only that properties are satisfied by most points in a given set.

Given an arbitrary direction ν, for most points $x\in\Omega'$ we have

$$\int|\nabla^2 u|(x+t\nu)\,dt\le c\eta\qquad\text{and}\qquad\int\mathrm{dist}(\nabla u,K)(x+t\nu)\,dt\le c\varepsilon,$$

where both integrals are extended to the $t\in\mathbb{R}$ such that $x+t\nu\in\Omega$. The first condition implies that there is $F\in\mathbb{R}^{2\times2}$ such that $|\nabla u-F|\le c\eta$ for any t. The second implies that this F is ε-close to K, let us assume it to be close to $SO(2)A$. If η is sufficiently small, the oscillation of ∇u is not sufficient to cover half of the distance between the two wells. For small ε and η we have $\mathrm{dist}(\nabla u, SO(2)A) \le \mathrm{dist}(\nabla u, K)$ on the considered segment. Thus, on such 'good' lines only one well is used, and we obtain

$$\int \mathrm{dist}(\nabla u, SO(2)A)(x+t\nu)dt \le c\varepsilon\,.$$

We may assume $A = \mathrm{Id}$ by a change of variables. The key point is now that u can only shorten distances along ν, up to errors of order ε. In particular,

$$\begin{aligned}
|u(x+t_2\nu) - u(x+t_1\nu)| &\le \int_{t_1}^{t_2} |\partial_\nu u|(x+t\nu)dt \\
&\le |t_1 - t_2| + \int_{t_1}^{t_2} \mathrm{dist}(\nabla u, SO(2))dt \\
&\le |t_1 - t_2| + c\varepsilon\,.
\end{aligned}$$

The same argument applied to the inverse function u^{-1} gives the converse bound (with the notation $\tilde{t} = x + t\nu$)

$$|t_1 - t_2| = \left|u^{-1}(u(\tilde{t}_1)) - u^{-1}(u(\tilde{t}_2))\right| \le |u(\tilde{t}_1) - u(\tilde{t}_2)| + c\varepsilon$$

(in the full proof this step becomes complex, since in general u does not have an inverse). The result is that we find many segments whose length is approximately unchanged by u.

Consider now a triangle $[abc]$, whose sides are rigid in the sense above (we denote by $[abc]$ the triangle with vertices a, b and c). Then the map u restricted to the three points a, b and c is close to an isometry, in the sense that the affine interpolation between the three values has gradient close to $O(2)$. For simplicity we assume it be the close to the identity. We get $|u(a) - a| \le c\varepsilon$, and the same for b and c. Let p be any point on the boundary of the triangle, say $p \in [a, b]$. By the previous argument the length of $u([a, p])$ is at most ε more than $|u(a) - u(p)|$, hence

$$|u(a) - u(p)| \le |a - p| + c\varepsilon\,,$$

and the same for b. This implies that $u(p)$ is at most a distance $\varepsilon^{1/2}$ away from the corresponding point in the segment $[u(a), u(b)]$ (and hence from p), see Figure 2.2. The same holds for all points on the boundary of $[abc]$. We now assume in addition that there exists $\nu \in \mathbb{R}^2$, $|\nu| = 1$, such that $|B\nu| < 1$, and use $(\nu, \nu^\perp)$ as a basis for $\mathbb{R}^2$. We fix $s \in \mathbb{R}$ such that the line $t \to t\nu + s\nu^\perp$ intersects $[abc]$ along a segment, and denote its extrema by t_1 and t_2. Consider now

$$u(t_1 + s\nu^\perp) - u(t_2 + s\nu^\perp) = \int_{t_1}^{t_2} \partial_\nu u(t\nu + s\nu^\perp)dt\,. \tag{2.4}$$

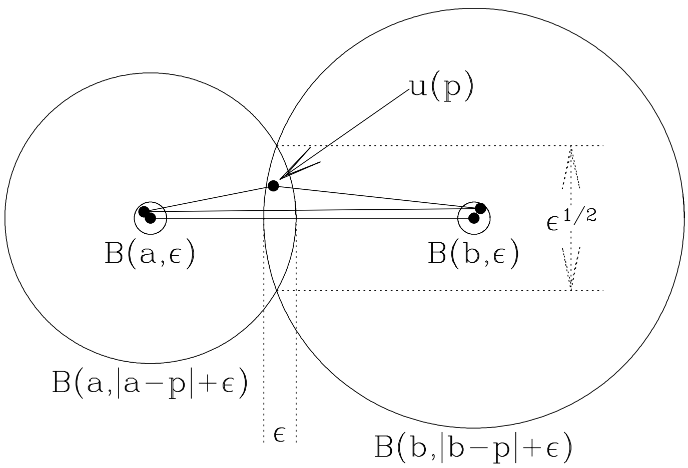

FIGURE 2.2: Points belonging to a 'rigid' segment ab can move only to order ε in the tangential direction and to order $\varepsilon^{1/2}$ in the normal direction. The extrema obey $u(a) \in B(a, c\varepsilon)$, and the same for b. The point $u(p)$ is in $B(a, |a-p| + c\varepsilon) \cap B(b, |b-p| + c\varepsilon)$. Hence $|u(p) - p| \le c\varepsilon^{1/2}$, and the projection of $u(p) - p$ along $a - b$ is controlled by $c\varepsilon$. In the figure labels the constants are not indicated explicitly.

The left-hand side has, by the previous argument, length $|t_2 - t_1| + O(\varepsilon^{1/2})$. To estimate the right-hand side, let E denote the set where the minority phase is used, i.e. the set where $\operatorname{dist}(\nabla u, SO(2)B) < \operatorname{dist}(\nabla u, SO(2))$. Then, we have pointwise

$$|\partial_\nu u| \le 1 + (|B\nu| - 1)\chi_E + \operatorname{dist}(\nabla u, K)\,,$$

and (2.4) gives

$$|t_1 - t_2| - c\varepsilon^{1/2} \le |t_1 - t_2| + \int_{t_1}^{t_2} [\operatorname{dist}(\nabla u, K) + (|B\nu| - 1)\chi_E]\, dt\,.$$

Integrating over s, one gets (recall that $|B\nu| < 1$)

$$\int_{[abc]} \chi_E\, dx \le c\varepsilon^{1/2} + c\int_{[abc]} \operatorname{dist}(\nabla u, K)\, dx \le c\varepsilon^{1/2} + c\varepsilon\,,$$

which, up to the exponent in the first error term, implies the desired estimate. The exponent is finally raised to one by considering two neighbouring triangles and lines that start or end in vertices (for which the length is controlled up to order ε), see Figure 2.5 and Lemma 2.3 below.

If instead $|B\nu| \ge |\nu|$ for all $\nu \in \mathbb{R}^2$, then either $B \in SO(2)$ an there is nothing to prove, or $\det B > \det A$. In the latter case

$$\int_{[abc]} \det \nabla u\, dx \ge |[abc]| + (\det B - 1)\,|E \cap [abc]| - c\varepsilon$$

and by the previous argument

$$\int_{[abc]} \det \nabla u \, dx = |u\left([abc]\right)| \leq |[abc]| + c\varepsilon^{1/2} .$$

Comparing the two expressions we conclude $|E \cap [abc]| \leq c\varepsilon^{1/2}$.

2.2 Segment rigidity

We now present a precise statement of the fact that one can find many segments whose length is approximately preserved. This segment-rigidity will also be the main ingredient of the $H^{1/2}$ estimate in Section 3.3, hence we state and prove it in slightly more general form than needed for Theorem 2.1. In particular we only assume a weaker control of the surface energy. For notational simplicity we formulate the statement only for the case $A = \mathrm{Id}$ (which is the only one used in the rest of this paper), the general form can be obtained with a change of variables. Further, a direct extension to multiwell energies is possible, with the same proof and only notational changes in the statement. The geometry is illustrated in Figure 2.3.

Proposition 2.2. *Let $\alpha \in (0, 1/8)$, $\theta \in (0, 1)$, $\bar{c} > 0$, $p \geq 1$, and $B \in \mathbb{R}^{2\times 2} \setminus SO(2)$. Then there are η and c (depending only on the above) such that the following holds. Let $u \in C^1(\Omega, \mathbb{R}^2)$ satisfy, for some $r > 0$,*

$$\frac{1}{r^2} \int_\Omega \phi(\nabla u) dx + \frac{1}{r} \int_\Omega |\nabla \phi(\nabla u)| \leq \eta \tag{2.5}$$

where $\phi(F) \geq \bar{c}\, \mathrm{dist}^p(F, SO(2))$, and let x_0, y_0 be such that $r < |x_0 - y_0| < 2r - 4\alpha r$, with $B((x_0 + y_0)/2, r) \subset \Omega$. Then, there is a subset of $(x, y) \in B(x_0, \alpha r) \times B(y_0, \alpha r)$ with measure at least $1 - \theta$ the total such that

$$1 - c\varepsilon \leq \frac{|u(x) - u(y)|}{|x - y|} \leq 1 + c\varepsilon \tag{2.6}$$

and

$$\frac{1}{r} \int_{[x,y]} \mathrm{dist}(\nabla u, SO(2)) d\mathcal{H}^1 \leq c\varepsilon \tag{2.7}$$

where

$$\varepsilon = \frac{1}{r^2} \int_{B_r} \mathrm{dist}(\nabla u, SO(2) \cup SO(2)B) dx . \tag{2.8}$$

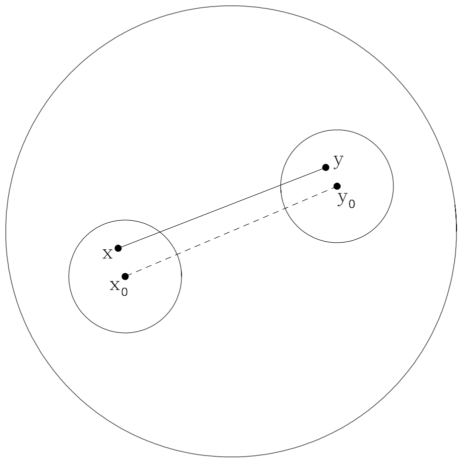

FIGURE 2.3: Sketch of the geometry in the statement of Proposition 2.2. The two points x_0 and y_0 are fixed, and there are two small balls around them such that they are well separated from each other and from the boundary of the domain. Then, for most pairs (x, y), the segment $[xy]$ is rigid under u.

Proof. The proof is based on obtaining from (2.5) a qualitative control on the behavior of u on a large part of the domain which is sufficient to derive an estimate for the desired quantities in terms of ε (in the relevant cases, ε is much smaller than η).

We divide the proof in several steps. First we define a 'bad' set Ω_{LG} of points where ∇u is far from a rotation, and show that Ω_{LG} is small and has small perimeter. By continuity, the gradient is still good on the boundary of the 'bad' set, hence the image of its boundary is also small. This shows that for a large fraction of the possible (x, y) both segments $[x, y]$ and $[u(x), u(y)]$ do not intersect the 'bad' set (or its image). Then we show that we can invert u along $[u(x), u(y)]$, to obtain a smooth curve γ joining x with y, with energy of order ε and away from the 'bad' set. A straightforward line integration concludes the proof. Without loss of generality we can assume $r = 1$ (by scaling), $x_0 + y_0 = 0$ (by a translation), and $\Omega = B(0, 1)$ (restricting u).

Step 1. Definition and size of the bad set. By the coarea formula for BV functions [2], from (2.5) we obtain

$$\eta \geq |\nabla\phi(\nabla u)|(\Omega) = \int_{\mathbb{R}} \mathrm{Per}_\Omega(\{x \in \Omega : \phi(\nabla u(x)) \geq t\})dt\,.$$

Therefore for any fixed $c_1 > 0$ there is $c_2 \in (c_1/2, c_1)$ such that the set

$$\Omega_{LG} = \{x \in \Omega : \phi(\nabla u(x)) \geq c_2\}$$

satisfies the bound

$$\mathrm{Per}_\Omega(\Omega_{LG}) \le \frac{1}{c_1}\eta\,.$$

We choose c_1 small enough so that $\phi(F) \le c_1$ implies $\mathrm{dist}(F, SO(2)) < \min(\mathrm{dist}(F, SO(2)B), 1/4)$. By (2.5) we then get $|\Omega_{LG}| \le c\eta$.

The gradient ∇u is uniformly close to $SO(2)$ on $\Omega\backslash\Omega_{LG}$, and by continuity also on the boundary. Therefore $u(\partial\Omega_{LG})$ is rectifiable and

$$\mathcal{H}^1\left(u(\Omega\cap\partial\Omega_{LG})\right) \le c\eta\,.$$

Further, $u(\Omega_{LG})$ has small area. Indeed,

$$\begin{aligned} |u(\Omega_{LG})| &\le \int_{\Omega_{LG}} |\det\nabla u| dx \le \int_{\Omega_{LG}} (1+\mathrm{dist}(\nabla u, SO(2)))^2\, dx \\ &\le c|\Omega_{LG}| + c\int_\Omega \phi^{2/p}(\nabla u) dx \le c\eta\,. \end{aligned} \tag{2.9}$$

where in the last step we used by the embedding of BV into L^2 and (2.5).

Step 2. Degree and local injectivity. We now show that there is a ball B' slightly smaller than $\Omega = B(0,1)$ such that on a large part of B' the function u is invertible. The key fact is that u has degree one, since it is approximately rigid on the boundary of B'.

From the quantitative rigidity estimate by Friesecke, James and Müller [21] (see also (2.31) below) it follows that, for some $Q\in SO(2)$,

$$\int_\Omega |\nabla u - Q|^2 dx \le c\int_\Omega \mathrm{dist}^2(\nabla u, SO(2)) dx \le c\left(\int_\Omega \phi^2(\nabla u)dx\right)^{1/p} \le c\eta^{2/p}\,,$$

where in the last step we exploited again the embedding of BV in L^2, and by the Poincaré inequality u is close to an isometry $I(x) = Qx+b$,

$$\int_\Omega |\nabla u - \nabla I|^2 + |u-I|^2\, dx \le c\eta^{2/p}\,. \tag{2.10}$$

Therefore there is $r' \in (1-\alpha/2, 1)$ such that

$$\int_{\partial B'} |\nabla u - \nabla I|^2 + |u-I|^2\, d\mathcal{H}^1 \le c\eta^{2/p}\,,$$

where $B' = B(0,r')$. By the embedding of $W^{1,2}(\partial B')$ in $L^\infty(\partial B')$ we have

$$|u(x) - I(x)| < c_3\eta^{1/p} \qquad \text{for all } x\in\partial B'\,. \tag{2.11}$$

Consider now the homotopy defined for $t \in [0,1]$ by

$$v_t(x) = tu(x) + (1-t)I(x)\,.$$

The map v_t is C^1, $v_0 = I$ and $v_1 = u$. By (2.11) the maps I and u differ on $\partial B'$ by less than $c_3\eta^{1/p}$. Let $B'' = B(0, r' - \alpha/2)$, so that $B(x_0, \alpha) \subset B''$, and the same for y_0. Then, provided η is small enough, by (2.11) for any t

$$v_t(\partial B') \cap I(B'') = \emptyset$$

and the Brouwer degree $\deg(v_t, B', z)$ does not depend on t, for any $z \in I(B'')$ (see e.g. [18, Theorem 1.12]). The isometry I has degree one, hence

$$\deg(u, B', z) = 1 \qquad \text{for all} \quad z \in I(B'')\,.$$

This implies that

$$\text{all } z \in I(B'') \setminus u(\Omega_{LG}) \text{ have exactly one } x \in B' \cap u^{-1}(z)\,. \tag{2.12}$$

Indeed, since $z \notin u(\Omega_{LG})$ all points in $B' \cap u^{-1}(z)$ have $\det \nabla u > 0$. But the degree being one, there is exactly one of them.

Consider now the set

$$\Omega_{NI} = \{x \in B' : u(x) \in u(\Omega_{LG}) \cap I(B'')\} = B' \cap u^{-1}(u(\Omega_{LG}) \cap I(B''))\,.$$

Away from Ω_{LG} we have $\det \nabla u \geq 1/2$, hence

$$\begin{aligned} |\Omega_{NI}| &\leq |B' \cap \Omega_{LG}| + \int_{B' \setminus \Omega_{LG}} \chi_{\Omega_{NI}} dx \\ &\leq |\Omega_{LG}| + 2 \int_{u(\Omega_{LG}) \cap I(B'')} \#(u^{-1}(z) \cap B' \setminus \Omega_{LG}) dz\,. \end{aligned}$$

But for $z \in I(B'')$ we have $\#(u^{-1}(z) \cap B' \setminus \Omega_{LG}) \leq 1 + \#(u^{-1}(z) \cap B' \cap \Omega_{LG})$, since the degree is one and $\det \nabla u > 0$ away from Ω_{LG}. Further,

$$\int_{u(\Omega_{LG})} \#(u^{-1}(z) \cap \Omega_{LG})\, dz = \int_{\Omega_{LG}} |\det \nabla u|\, dx \leq c\eta$$

by (2.9). This implies

$$|\Omega_{NI}| \leq c\eta\,. \tag{2.13}$$

Step 3. Choice of x and y. In the following we show that several properties are satisfied by a large fraction of the possible choices of x and y in the balls $B(x_0, \alpha)$ and $B(y_0, \alpha)$. The quantity η in the statement is chosen so small that the 'bad' choices for each of the considered properties are less that $\theta/10$ of the possible choices, and since we consider less than 10 properties this shows that for a $1-\theta$ fraction of the choices all properties are satisfied. To shorten notation, we say that for most choices property P holds if

$$\begin{aligned}\mathcal{H}^4\left(\{(x,y) \in B(x_0,\alpha) \times B(y_0,\alpha) : P \text{ does not hold}\}\right) \\ \leq \frac{\theta}{10}\mathcal{H}^4\left(B(x_0,\alpha) \times B(y_0,\alpha)\right) .\end{aligned}$$

First, since $|\Omega_{LG}| \leq c\eta$ we have that, if η is small enough, for most choices

$$x \notin \Omega_{LG} \qquad \text{and} \qquad y \notin \Omega_{LG} . \tag{2.14}$$

In the sequel we work in the reduced domains $B_1 = B(x_0, \alpha) \setminus \Omega_{LG}$ and $B_2 = B(y_0, \alpha) \setminus \Omega_{LG}$, which for sufficiently small η satisfy

$$|B_1| \geq c\alpha^2 , \quad |B_2| \geq c\alpha^2 , \quad |u(B_1)| \geq c\alpha^2 , \quad |u(B_2)| \geq c\alpha^2 .$$

We now show that for most pairs (x, y)

$$\int_{[x,y]} \mathrm{dist}(\nabla u, SO(2)) d\mathcal{H}^1 \leq c\varepsilon , \tag{2.15}$$

where c depends on α and θ. We first consider the integral of $\mathrm{dist}(\nabla u, K)$. Let $f = \mathrm{dist}(\nabla u, K)$ in Ω, and zero elsewhere, so that the integral over $[x, y]$ can be extended to the entire line joining x with y. Then, we integrate the result over $(x, y) \in B_1 \times B_2$, and change variables. Let $x \in B_1$, $\nu \in S^1$, and consider the line $t \to x + t\nu$. Only those lines for which $|\nu - \nu_0| \leq \alpha$ can intersect B_2 (see Figure 2.4, here $\nu_0 = (y_0 - x_0)/|y_0 - x_0|$). The Jacobian of the transformation is uniformly bounded, since the radius α of the two balls is smaller than $|x_0 - y_0|/4$. This gives

$$\begin{aligned}\int_{B_1\times B_2} dx\, dy \int_{[x,y]} f\, d\mathcal{H}^1 \;&\leq\; c\alpha \int_{|\nu-\nu_0|\leq\alpha} d\nu \int_{B_1} dx \int_{\mathbb{R}} dt\, f(x+t\nu) \\ &\leq\; c\alpha^3 \int_\Omega f\, dx \leq c\alpha^3\varepsilon ,\end{aligned}$$

and implies that for most pairs (x, y) we have $\int_{[x,y]} f\, d\mathcal{H}^1 \leq c\varepsilon$.

Now consider the set $\partial\Omega_{LG}$. For any fixed $\nu \in S^1$ the set A_ν of $x \in B_1$ such that the line $t \to x + t\nu$ intersects $\partial\Omega_{LG}$ has area smaller than

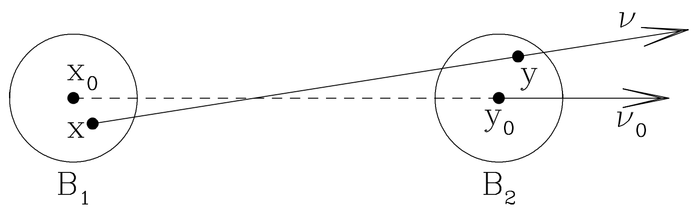

FIGURE 2.4: The balls B_1 and B_2 around x_0 and y_0 are well separated compared to their radius α. Therefore all lines which intersect both have direction close to the one joining their centers. For each $x \in B_1$, we write the integral over $y \in B_2$ in polar coordinates, with center x. See text.

$2\alpha\mathcal{H}^1(\partial\Omega_{LG}) \le c\alpha\eta$. To see this, let $x = x_0 + s\nu + r\nu^\perp \in A_\nu$. Then $|s| \le \alpha$, and $x_0 + r\nu^\perp$ is contained in the orthogonal projection of $\partial\Omega_{LG}$ on the line through x_0 parallel to $\nu^\perp$. Therefore the $\mathcal{H}^1$ measure of the set of possible r is less than $\mathcal{H}^1(\partial\Omega_{LG})$. This proves the claim, which in turn implies that for most pairs the segment $[x, y]$ does not intersect $\partial\Omega_{LG}$. Recalling (2.14),

$$[x, y] \cap \Omega_{LG} = \emptyset\,.$$

This implies $\mathrm{dist}(\nabla u, SO(2)) \le f = \mathrm{dist}(\nabla u, K)$ on $[x, y]$, hence (2.15).

Now we consider analogously the segments $[u(x), u(y)]$ in the image. By (2.13) for most choices of (x, y) one has

$$u(x), u(y) \notin u(\Omega_{LG})\,.$$

We claim that for most choices of x and y one also has

$$[u(x), u(y)] \cap u(\partial\Omega_{LG}) = \emptyset\,. \tag{2.16}$$

To see this, we focus on the smaller domains

$$\tilde{B}_i = \left\{ x \in B_i \setminus \Omega_{NI} : |u(x) - I(x)| \le \frac{1}{2}\alpha \right\}\,, \qquad i = 1, 2\,.$$

By (2.10) and (2.13) we have $|B_i \setminus \tilde{B}_i| \le c\eta^{1/p}$. Now we repeat the previous argument in the image. For any $\xi \in u(\tilde{B}_1) \subset B(I(x_0), \frac{3}{2}\alpha)$ the lines $t \to \xi + t\nu$ can intersect $u(\tilde{B}_2) \subset B(I(y_0), \frac{3}{2}\alpha)$ only if $|\nu - I(\nu_0)| \le 3\alpha$. Fix one such ν and consider

$$A_\nu = \left\{ \xi \in u(\tilde{B}_1) : \{x + t\nu\}_{t\in\mathbb{R}} \cap u(\partial\Omega_{LG}) \ne \emptyset \right\}.$$

Reasoning as above, we get $|A_\nu| \le c\mathcal{H}^1(u(\partial\Omega_{LG})) \le c\eta$, hence (2.16) follows. All segments $S = [u(x), u(y)]$ with $(x, y) \in \tilde{B}_1 \times \tilde{B}_2$ are contained in $I(B'')$, hence $S \subset u(B')$. Even more, $S \subset u(B' \setminus \Omega_{LG})$. Indeed, by continuity if this was not the case then $u(\partial\Omega_{LG})$ would intersect S, contradicting (2.16). Therefore by (2.12) we can invert u along S. The inverse is locally C^1, since away from Ω_{LG} the determinant is close to 1. It follows that there is a curve $\gamma_{xy} : [0,1] \to B'$ such that $u \circ \gamma_{xy}$ is a C^1 monotonic parametrization of the segment $[u(x), u(y)]$.

It remains to show that for most (x, y) the curve γ_{xy} carries energy of order ε. We define $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(z) = \sum_{x \in u^{-1}(z) \cap (B' \setminus \Omega_{LG})} \mathrm{dist}(\nabla u, K)(x)\,,$$

where it is understood that $g = 0$ if the sum is empty. It is clear that

$$\int_{\mathbb{R}^2} g\, dz = \int_{B' \setminus \Omega_{LG}} f \det \nabla u\, dx \le c\varepsilon\,.$$

As above, the set of $(\xi, \eta) \in B(I(x_0), \frac{3}{2}\alpha) \times B(I(y_0), \frac{3}{2}\alpha)$ where

$$\int_{[\xi,\eta]} g\, d\mathcal{H}^1 \le c\varepsilon$$

does not hold, is small. Then, also the set of $(x, y) \in \tilde{B}_1 \times \tilde{B}_2$ where the same condition over $[u(x), u(y)]$ is violated is small. Hence for most pairs (x, y)

$$\int_{\gamma_{xy}} f\, d\mathcal{H}^1 \le c \int_{[u(x),u(y)]} g\, d\mathcal{H}^1 \le c\varepsilon\,. \tag{2.17}$$

Step 4. Length estimates. We have shown that

$$\int_{[x,y]} \mathrm{dist}(\nabla u, SO(2))\, d\mathcal{H}^1 \le c\varepsilon \quad \text{and} \quad \int_{\gamma_{xy}} \mathrm{dist}(\nabla u, SO(2))\, d\mathcal{H}^1 \le c\varepsilon\,,$$

where γ_{xy} is a C^1 curve joining x and y, with $u \circ \gamma_{xy}$ being a monotonic parametrization of the segment $[u(x), u(y)]$. The first condition implies

$$\begin{aligned} |u(x) - u(y)| &\le \int_{[x,y]} |\nabla_\tau u|\; d\mathcal{H}^1 \\ &\le |x - y| + \int_{[x,y]} \mathrm{dist}(\nabla u, SO(2)) d\mathcal{H}^1 \\ &\le |x - y| + c\varepsilon\,, \end{aligned}$$

where ∇_τ denotes the tangential derivative. Since u is a one-to-one map of the curve γ_{xy} onto the segment $[u(x), u(y)]$,

$$\begin{aligned} |u(x)-u(y)| &= \int_{\gamma_{xy}} |\nabla_\tau u| \, d\mathcal{H}^1 \\ &\geq \mathcal{H}^1(\gamma_{xy}) - \int_{\gamma_{xy}} \mathrm{dist}(\nabla u, SO(2)) d\mathcal{H}^1 \\ &\geq |x-y| - c\varepsilon \, . \end{aligned}$$

This concludes the proof. □

2.3 Two-well rigidity in L^1

We now come to the second main ingredient of the proof of Theorem 2.1, namely, a construction showing that if the sides of two appropriately chosen triangles are rigid, then only one phase is used, up to a small error.

Lemma 2.3. *Let A and $B \in \mathbb{R}^{2\times 2}$ have positive determinant. Then there are positive numbers ρ, c, η depending only on A and B such that for any $u : Q = (-r, r)^2 \to \mathbb{R}^2$ which obeys*

$$\frac{1}{r} \int_Q \left|\nabla^2 u\right| \leq \eta \tag{2.18}$$

for some $r > 0$, one has

$$\min_{J\in\{A,B\}} \int_{Q'} \mathrm{dist}(\nabla u, SO(2)J) dx \leq c \int_Q \mathrm{dist}(\nabla u, K) dx, \tag{2.19}$$

where $Q' = (-\rho r, \rho r)^2$ and $K = SO(2)\{A, B\}$.

Proof. By scaling we can assume $r = 1$, and by density that u is smooth. We denote by E the set where the B-phase is used,

$$E = \{x \in Q : \mathrm{dist}(\nabla u, SO(2)B) \leq \mathrm{dist}(\nabla u, SO(2)A)\} \tag{2.20}$$

and assume it is the minority one, $|E| \leq |Q|/2$ (if not, we swap A and B). This implies

$$\int_{Q\setminus E} \mathrm{dist}(\nabla u, SO(2)A) dx \leq \int_Q \mathrm{dist}(\nabla u, K) dx = \varepsilon \, ,$$

where the last equality defines ε. By the Poincaré estimate, (2.18) implies that there is $F \in \mathbb{R}^{2\times 2}$ such that

$$\int_Q |\nabla u - F| dx \leq c\eta \, , \tag{2.21}$$

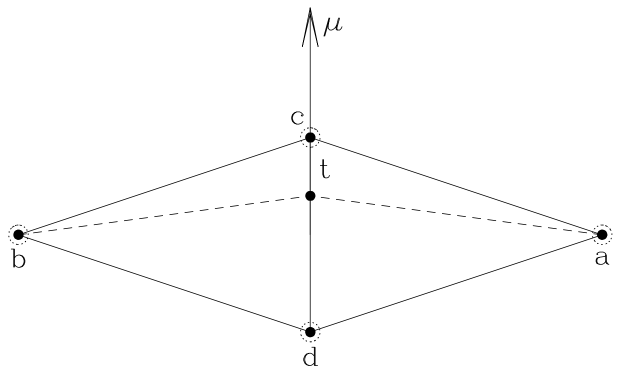

FIGURE 2.5: Sketch of the construction used in Lemma 2.3. The points $u(a)$, $u(b)$, $u(c)$ and $u(d)$ are located in ε-balls around a, b, c and d respectively.

and by the previous estimate $\operatorname{dist}(F, SO(2)A) \leq c\eta + c\varepsilon$. If $\varepsilon \geq \eta$ the proof is concluded; in the following we will therefore assume $\varepsilon \leq \eta$.

The rest of the proof is divided in two cases, depending on whether there is a vector ξ such that $|B\xi| < |A\xi|$, or not.

Case 1: $|B\xi| < |A\xi|$ for some $\xi \in \mathbb{R}^2$. We set $\phi(G) = \operatorname{dist}(G, SO(2)A)$, for $G \in \mathbb{R}^{2\times 2}$. Then (2.18) and (2.21) give

$$\int_Q |\phi(\nabla u)|dx + \int_Q |\nabla\phi(\nabla u)| \leq c\eta\,. \tag{2.22}$$

We change variables according to $\tilde{u}(x) = u(A^{-1}Qx)$, for a suitable $Q \in SO(2)$, and reduce to the case $A = \mathrm{Id}$ and $|Be_1| < 1$. This change of variables only depends on A and B, hence all constants in the statement are affected only by factors depending on A and B. Let $\delta > 0$ be such that

$$|B\xi| < 1 - 2\delta \qquad \text{for all } \xi \in \mathbb{R}^2 \text{ s.t. } |\xi| = 1,\ |\xi - e_1| < 2\delta\,.$$

Consider the rhombus with vertices

$$a_0 = \left(\frac{1}{2}, 0\right) \quad b_0 = \left(-\frac{1}{2}, 0\right) \quad c_0 = (0, \delta) \quad d_0 = (0, -\delta)$$

(see Figure 2.5). Now we fix two small radii $\rho \ll \tilde{\rho} \ll \delta$ (to be chosen later), and claim that there are

$$c \in B(c_0, \rho)\,, \quad d \in B(d_0, \rho)\,,$$

such that for many symmetric choices of

$$a \in B(a_0, \tilde{\rho})\,, \quad b \in B(b_0, \tilde{\rho})\,,$$

the five segments $[a,c]$, $[a,d]$, $[b,c]$, $[b,d]$, $[c,d]$ satisfy the statement of Proposition 2.2. By symmetric we mean that a and b are symmetric with respect to the line joining c with d. Furthermore, all points can be chosen so that

$$|u(x)-(Fx+q)| \le \frac{1}{10}\delta\,, \qquad \text{for } x \in \{a,b,c,d\}\,, \tag{2.23}$$

with a unique $q \in \mathbb{R}^2$. The latter property is clearly satisfied by most choices in the given balls, since by (2.21) there is a q such that $\int |u(x)-(Fx+q)|dx \le c\eta$. If the radii ρ and $\tilde\rho$ are sufficiently small compared to δ, then for any $p \in [c,d]$ we have $|B(a-p)| < |a-p|(1-\delta)$ and $|B(b-p)| < |b-p|(1-\delta)$.

The claim is proved based on Proposition 2.2. For each pair $\{x,y\} \subset \{a_0,b_0,c_0,d_0\}$ there are many choices which are 'good'. Fix a small $\zeta > 0$ and consider the set

$$\begin{aligned} S_{ca} \;=\; \{c \in B(c_0,\rho) : \ &\text{for at least } 1-\zeta \text{ of the } a \in B(a_0,\rho) \\ &\text{the segment } [c,a] \text{ is as given in Proposition 2.2}\} \end{aligned}$$

and analogously S_{cb}, S_{cd}. By Proposition 2.2, if η is small enough each of them covers at least a $(1-\zeta)$-fraction of $B(c_0,\rho)$. Therefore their intersection $S_c = S_{ca} \cap S_{cb} \cap S_{cd}$ covers at least a $(1-3\zeta)$-fraction of the ball.

Now consider the point d. Analogously, there is a set S_d covering most of $B(d_0,\rho)$ such that for each such d, most c in S_c, most a in $B(a_0,\rho)$ and most b in $B(b_0,\rho)$ will give good segments (note that we require the first segment to end in some $c \in S_c$). Pick now any $d \in S_d$, and any of the corresponding $c \in S_c$. Then it is immediate that only few choices of a and b are not admissible. Let T_a be the set of bad a, and T_b the set of bad b (both are small). Let P_{cd} be the map representing symmetry across the line through c and d. Since $\tilde\rho \gg \rho$, $P_{cd}B(a_0,\tilde\rho)$ overlaps $B(b_0,\tilde\rho)$ over a large area, but the union of the 'bad' points $P_{cd}T_a \cup T_b$ is small. Hence there are many good choices of symmetric a and b. This concludes the proof of the claim.

Proposition 2.2 yields that the five lengths are preserved by u, up to errors of order ε. This implies that there is an isometry $x \to Qx+p$ such that

$$|u(x)-(Qx+p)| \le c\varepsilon \qquad \text{for } \ x \in \{a,b,c,d\}\,. \tag{2.24}$$

Note that in a first step we possibly find two different isometries for the two triangles. By (2.23) both isometries have positive orientation, and since the triangles have an edge in common we can assume that the isometries are the same. For simplicity of notation we assume without loss of generality that $Q = \mathrm{Id}$ and $p = 0$. Then,

$$|u(a)-a| \le c\varepsilon\,, \qquad \int_{[a,c]} \mathrm{dist}(\nabla u, SO(2))d\mathcal{H}^1 \le c\varepsilon\,,$$

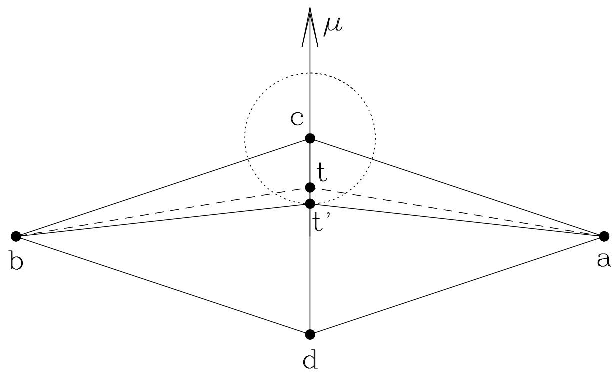

FIGURE 2.6: Location of $u(t)$ in the proof of Lemma 2.3. The minimum of $|a-t'|+|b-t'|$, among all t' in a given circle centered in c, is achieved by the point of the circle which belongs to the segment $[c,d]$ and is closest to its midpoint $(c+d)/2$. We use here that a and b are symmetric with respect to the line cd, and that t is closer to c than to d.

and the same for the other 3 points and the other 4 segments.

Let t be any point of $[c,d]$. Assume for definiteness that $|c-t| \le |d-t|$, in the other case we proceed analogously using d instead of c. The estimate (2.7) of Proposition 2.2 gives

$$|u(c)-u(t)| \le |c-t| + \int_{[c,t]} \mathrm{dist}(\nabla u, SO(2)) d\mathcal{H}^1 \le |c-t| + c\varepsilon\,.$$

Combining with the previous estimate on $|u(c)-c|$ we get $|c-u(t)| \le |c-t|+c\varepsilon$, which in turn implies

$$|a-u(t)|+|b-u(t)| \ge |a-t|+|b-t|-c\varepsilon\,,$$

see Figure 2.6. Since $|u(a)-a| \le c\varepsilon$ and the same for b, we also get

$$|u(a)-u(t)|+|u(b)-u(t)| \ge |a-t|+|b-t|-c\varepsilon\,. \tag{2.25}$$

Consider now the segments $[a,t]$ and $[b,t]$. Since along those directions the matrix B is strictly short (i.e. it maps each vector parallel to $a-t$ or $b-t$ to a shorter one), we get

$$|u(a)-u(t)| \le |a-t| + \int_{[a,t]} \mathrm{dist}(\nabla u, K) d\mathcal{H}^1 - \delta \int_{[a,t]} \chi_E\, d\mathcal{H}^1$$

and the same for $u(b)-u(t)$. Here χ_E denotes the characteristic function of the set E where the minority phase is used, defined in (2.20). Comparing with (2.25) we get

$$\int_{[a,t]\cup[t,b]} \chi_E\, d\mathcal{H}^1 \le \delta^{-1} \int_{[a,t]\cup[t,b]} \mathrm{dist}(\nabla u, K) d\mathcal{H}^1 + c\delta^{-1}\varepsilon\,. \tag{2.26}$$

The factor $1/\delta$ can be included in the constants, since it only depends on A and B. We integrate over all $t \in [c,d]$ and change variables to obtain an integration over the rhombus $R = [adbc]$,

$$\int_R \chi_E \, J \, dx \le c \int_R \mathrm{dist}(\nabla u, K) \, J \, dx + c\varepsilon \,. \tag{2.27}$$

Here $J(x)$ is the Jacobian determinant, which behaves as $1/\mathrm{dist}(x, \{a,b\}) \ge 1$. Since this is bounded from below, in the left-hand side we can simply drop J. To show that the integral in the right-hand side is bounded by ε we make use of the remaining freedom in the choice of a and b, and of the fact that the integral of J over the rhombus is finite. Precisely, we enlarge the integral to a symmetric rhombus with a 90-degree aperture, and sum the contributions on the two sides. Let

$$g(x) = \mathrm{dist}(\nabla u(x), K) + \mathrm{dist}(\nabla u(P_{cd}x), K) \,,$$

where as always we extend by zero outside the domain, and

$$C_a = \{x : 0 \le x_1 \le a_1 \,, \; |x_2 - a_2| \le a_1 - x_1\}$$

be the translation of a fixed cone by a, and let $D_x = \{a : x \in C_a\}$ be the symmetric cone. We obtain

$$\int_{\mathbb{R}^2} da \int_{C_a} g(x) \frac{1}{|x-a|} dx = \int_{\mathbb{R}^2} dx \, g(x) \int_{D_x} \frac{1}{|x-a|} da \le c\varepsilon \,,$$

which shows that for most choices of a

$$\int_R \mathrm{dist}(\nabla u, K) \, J \, dx \le \int_{C_a} g \, J \, dx \le c\varepsilon \,.$$

This concludes the proof in the first case.

Case 2: $|B\xi| \ge |A\xi|$ for all $\xi \in \mathbb{R}^2$. Again, with a change variables we can assume $A = \mathrm{Id}$ and $\det B > 1$ (if $\det B \le 1$, then either we are in Case 1 or $B \in SO(2)$). By the usual choice argument based on Fubini, for one-half of the $r \in (1/2, 1)$ we have

$$\int_{\partial B_r} \mathrm{dist}(\nabla u, K) d\mathcal{H}^1 \le c\varepsilon \quad \text{and} \quad \int_{\partial B_r} \phi(\nabla u) + |\nabla \phi(\nabla u)| d\mathcal{H}^1 \le c\eta \,,$$

where $B_r = B(0,r)$. This implies that for most r the boundary ∂B_r is entirely in the A-phase, in the sense that we have $\mathrm{dist}(\nabla u, SO(2)) \le c\eta$ everywhere on ∂B_r. We fix one such r. The above gives, in particular,

$$\int_{\partial B_r} \mathrm{dist}(\nabla u, SO(2)) d\mathcal{H}^1 \le c\varepsilon \qquad \text{and} \qquad \det \nabla u \ge \frac{1}{2} \text{ on } \partial B_r \,.$$

The idea of the proof is to use the first of these estimates to control the change in length along ∂B_r, and then an isoperimetric inequality to show that the volume cannot increase by more then ε. The desired estimate will follow from $\det B > \det A = 1$.

To make this precise, we first observe that

$$\int_\Omega \det \nabla u \, dx = \int_{\mathbb{R}^2} \deg(u, \Omega, y) dy \, ,$$

as can be seen by a decomposition of the image in level sets of $\deg(u, \Omega, \cdot)$ or directly from the coarea formula. Now we apply the weighted isoperimetric inequality

$$\int_{\mathbb{R}^2} \deg(u, \Omega, y) dy \le \frac{1}{4\pi} \left(\int_{\partial\Omega} |\partial_\tau u| d\mathcal{H}^1 \right)^2 \, , \tag{2.28}$$

where ∂_τ denotes the tangential derivative (see below for a short proof), applied to $\Omega = B_r$. Since $|\partial_\tau u| \le 1 + \operatorname{dist}(\nabla u, SO(2))$ this gives

$$\int_{B_r} \det \nabla u \, dx \le \pi r^2 + c\varepsilon \, .$$

Let now E be the set where ∇u is in the minority (B-) phase. A straightforward computation shows that

$$\int_{B_r} \det \nabla u \, dx \ge |B_r| + (\det B - 1)|E| - c\varepsilon$$

which implies $|E| \le c\varepsilon$, hence the thesis.

We finally show how the weighted isoperimetric inequality (2.28) is derived from the standard isoperimetric inequality. We define, for $k \in \mathbb{Z}$,

$$\omega_k = \{ y \in \mathbb{R}^2 : \deg(u, \Omega, y) \ge k \} \, .$$

The boundaries $\partial\omega_k$ cover the jump set of $\deg(u, \Omega, \cdot)$, which in turn is contained in $u(\partial\Omega)$. We have

$$\int_{\mathbb{R}^2} \deg(u, \Omega, y) dy \le \sum_{k>0} |\omega_k| \le \frac{1}{4\pi} \sum_{k>0} |\partial\omega_k|^2 \le \frac{1}{4\pi} \left(\sum_{k>0} |\partial\omega_k| \right)^2 \, .$$

If the $\partial\omega_k$ are disjoint, up to an $\mathcal{H}^1$–null set, then

$$\sum_{k>0} |\partial\omega_k| \le \mathcal{H}^1 \left(u(\partial\Omega) \right) \le \int_{\partial\Omega} |\partial_\tau u| d\mathcal{H}^1$$

gives the desired (2.28). In the general case, we claim that

$$\text{for } \mathcal{H}^1\text{-a.e. } y \in u(\partial\Omega) \qquad \text{one has } \#\{k : y \in \partial\omega_k\} \le \#u^{-1}(y) \cap \partial\Omega\,. \tag{2.29}$$

Repeating the previous computation, this implies

$$\sum_{k>0} |\partial\omega_k| \le \int_{u(\partial\Omega)} \#\{k : y \in \partial\omega_k\} d\mathcal{H}^1 \le \int_{\partial\Omega} |\partial_\tau u| d\mathcal{H}^1$$

and hence (2.28). We finally prove (2.29). Fix some $y \in u(\partial\Omega)$. Since $u \in C^1(\bar\Omega)$ and $\det \nabla u \neq 0$ on $\partial\Omega$, the set $X = u^{-1}(y) \cap \partial\Omega$ is finite and for each $x_j \in X$ we can find a ball B_j such that u is a diffeomorphism on B_j. By continuity, there is a neighbourhood N of y whose counterimage is completely contained in the B_j's. Therefore

$$\deg(u, \Omega, z) = \sum_{j=1}^{N} \deg(u, \Omega \cap B_j, z) \qquad \text{for } z \in N\,.$$

The map u is a diffeomorphism in each B_j, hence the jump of each of the $\deg(u, \Omega \cap B_j, y)$ at y is ± 1. This shows that the jump of $\deg(u, \Omega, \cdot)$ at y is at most N and proves (2.29) (see e.g. [18] for the elementary properties of the degree used here). □

We finally come to the proof of Theorem 2.1, which is based on covering the domain by suitable squares and using Lemma 2.3 on each of them.

Proof of Theorem 2.1. Let $l = \mathrm{dist}(\Omega', \partial\Omega)$. The constant $c_2 = c_2(A, B)$ will be chosen in the second part of the proof. The Theorem is immediate if (2.3) does not hold. Indeed, by (2.1) and the Poincaré inequality

$$\int_\Omega |\nabla u - F| dx \le c_\Omega \eta_0 l$$

for some $F \in \mathbb{R}^{2\times 2}$. Therefore there are $J \in \{A, B\}$ and $Q \in SO(2)$ with

$$|\Omega|\; |F - QJ| = |\Omega| \,\mathrm{dist}(F, K) \le c_\Omega \eta_0 l + \int_\Omega \mathrm{dist}(\nabla u, K) dx\,.$$

This implies

$$\int_\Omega |\nabla u - QJ|\, dx \le 2 c_\Omega \eta_0 l + \int_\Omega \mathrm{dist}(\nabla u, K) dx$$

which, if (2.3) does not hold, is the thesis.

We now come to the interesting case, where (2.3) holds, i.e. $\int \mathrm{dist}(\nabla u, K)dx$ is small compared to l^2. By density we can assume $u \in C^1$. Now we are in a position to apply Lemma 2.3 to each sufficiently large square contained in Ω. The bound on the size of the square is given by the value of η. The result follows from a covering argument, which we give in some detail in order to trace the dependence of the constants on l.

Let ρ be as in Lemma 2.3. We can assume that $\rho \leq 1/4$. We fix $r = l\rho/2$, and let $x_1, \dots x_k$ be the points in $r\mathbb{Z}^2$ such that the squares $Q_i = x_i + (-r, r)^2$ intersect Ω'. The squares Q_i cover Ω', are contained in Ω, and each point $x \in \Omega$ is contained in at most 4 of them. Let now $r' = l/2$, and $Q_i' = x_i + (-r', r')^2$. Since $(r + r')\sqrt{2} = l(\rho + 1)/\sqrt{2} \leq l$, all $Q_i' \subset \Omega$. Choose now η_0 as one-half of the η in the statement of Lemma 2.3. Then (2.1) implies that Lemma 2.3 can be applied to each of the Q_i', hence for each i there is $J_i \in \{A, B\}$ with

$$\int_{Q_i} \mathrm{dist}(\nabla u, SO(2)J_i)dx \leq c^* \int_{Q_i'} \mathrm{dist}(\nabla u, K)dx\,, \tag{2.30}$$

where c^* only depends on A and B. We claim that J_i does not depend on i. Indeed, the squares Q_i cover the connected set Ω'. Therefore if $\{J_i\}$ were not constant there would be i and j such that $J_i = A$, $J_j = B$, $Q_i \cap Q_j \neq \emptyset$. Since the centers of the squares lie on $r\mathbb{Z}^2$, this implies $|Q_i \cap Q_j| \geq r^2$, and

$$\begin{aligned} r^2 \mathrm{dist}(A, SO(2)B) &\leq \int_{Q_i \cap Q_j} \mathrm{dist}(\nabla u, SO(2)J_i) + \mathrm{dist}(\nabla u, SO(2)J_j)dx \\ &\leq 2c^* \int_{Q_i \cup Q_j} \mathrm{dist}(\nabla u, K)dx \leq 2c^* c_2(A, B) l^2 \end{aligned}$$

where c^* is as in (2.30), and we used (2.3). But this is a contradiction, provided that c_2 is chosen small enough (precisely, we need $c_2 < \rho^2 c^* \mathrm{dist}(A, SO(2)B)/8$). Therefore J_i does not depend on i.

To conclude the proof, we just observe that

$$\begin{aligned} \int_{\Omega'} \mathrm{dist}(\nabla u, SO(2)J)dx &\leq \sum_i \int_{Q_i} \mathrm{dist}(\nabla u, SO(2)J)dx \\ &\leq c^* \sum_i \int_{Q_i'} \mathrm{dist}(\nabla u, K)dx \\ &\leq \frac{9c^*}{\rho^2} \int_{\Omega} \mathrm{dist}(\nabla u, K)dx\,, \end{aligned}$$

since any point on Ω belongs at most to $9/\rho^2$ of the Q_i'. $\square$

In closing this section we mention two consequences of our result.

Corollary 2.4. *Assumption (2.1) can be weakened to an analogous BV control of any scalar function which controls the distance from the wells. In particular, if A is the majority phase (i.e. $\operatorname{dist}(\nabla u, SO(2)A) \leq \operatorname{dist}(\nabla u, SO(2)B)$ on at least one-half of the domain), then one can replace (2.1) with*

$$\int_\Omega |\nabla \phi(\nabla u)| \leq \eta \operatorname{dist}(\Omega', \partial\Omega)$$

for some ϕ such that

$$\phi(F) \geq c \operatorname{dist}^p(F, SO(2)A) \qquad \text{for some } p \geq 1\,, \quad c > 0\,.$$

Proof. As above, exploiting the more general formulation of Proposition 2.2. □

Corollary 2.5. *Assume that*

$$\varepsilon = \int_\Omega \operatorname{dist}^p(\nabla u, K) dx$$

is small, and keep assumption (2.1) on the surface energy. Then,

$$\int_{\Omega'} \operatorname{dist}^p(\nabla u, SO(2)J) dx \leq c\varepsilon^{1/p}\,.$$

The exponent $1/p$ in this bound is optimal.

Proof. By estimating the L^1 norm with the L^p norm, Theorem 2.1 gives

$$\int_{\Omega'} \operatorname{dist}(\nabla u, SO(2)J) dx \leq c\varepsilon^{1/p}\,,$$

for some $J \in \{A, B\}$. The the result follows from

$$\operatorname{dist}^p(F, SO(2)J) \leq c\left[\operatorname{dist}^p(F, K) + \operatorname{dist}(F, SO(2)J)\right]\,.$$

The optimality of the exponent $1/p$ follows from a straightforward extension of the construction given in [13, Lemma 4.3]. □

2.4 One-well L^1 rigidity results

We briefly discuss here rigidity for the one-well problem in the L^1 case. The classical Liouville rigidity states that if $\nabla u \in SO(n)$ everywhere, then ∇u is a constant. A quantitative version of this result has recently been derived by Friesecke, James, and Müller [20, 21], who obtained that for any connected bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and any p strictly between 1 and ∞ there is a constant $c = c(p, \Omega)$ such that

$$\min_{Q\in SO(n)} \int_\Omega |\nabla u - Q|^p \, dx \le c \int_\Omega \mathrm{dist}^p(\nabla u, SO(n)) \, dx \tag{2.31}$$

for any $u : \Omega \to \mathbb{R}^n$. The proof makes use of a decomposition of u into a harmonic part and a remainder. Even if presented for the case $p = 2$, their proof can be extended with minor changes to the L^p case (see below). In the limiting case $p = \infty$ one only obtains an estimate in BMO, as was shown by John in [24]. We discuss here the case $p = 1$.

The corresponding linear version of the rigidity estimate, known as Korn's inequality, states that, for the same Ω and $1 < p < \infty$, there is a constant $c = c(p, \Omega)$ such that

$$\min_{S=-S^T} \|\nabla u - S\|_{L^p(\Omega)} \le c \left\|\nabla u + \nabla u^T\right\|_{L^p(\Omega)} \tag{2.32}$$

for any $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ [36, 38, 22]. Both (2.32) and (2.31) do not hold for $p = 1$, see [33, 11]. Indeed, for $p = 1$ one obtains the weaker estimate

$$\min_{S=-S^T} \|\nabla u - S\|_{\mathrm{w}-L^1(\Omega)} \le c \left\|\nabla u + \nabla u^T\right\|_{L^1(\Omega)}, \tag{2.33}$$

where the weak-L^1 quasinorm is defined by

$$\|f\|_{\mathrm{w}-L^1(\Omega)} = \sup_{\lambda>0} \lambda \left|\{x \in \Omega : \ |f(x)| \ge \lambda\}\right| .$$

It is clearly controlled by the L^1 norm, i.e. $\|f\|_{\mathrm{w}-L^1} \le \|f\|_{L^1}$, but not equivalent to it, as the example $f(x) = 1/x$ on the real line shows. The inequality (2.33) can be extended to functions of bounded deformation [1]. The combination of the Korn rigidity estimate with the Sobolev embedding, and the corresponding trace inequality, instead hold also in the critical case $p = 1$. Precisely, one has

$$\min_{S=-S^T,\, b\in\mathbb{R}^2} \|u(x)-Sx-b\|_{L^{1^*}(\Omega)} + \|u(x)-Sx-b\|_{L^1(\partial\Omega)} \le c \left\|\nabla u + \nabla u^T\right\|_{L^1(\Omega)}$$

where $1^* = n/(n-1)$, see [26, 3, 37].

We show here that analogous results hold in the nonlinear situation.

Proposition 2.6. *Let $v : \Omega \to \mathbb{R}^n$, with Ω bounded, Lipschitz and connected, and set*

$$\varepsilon = \|\mathrm{dist}(\nabla v, SO(n))\|_{L^1(\Omega)} .$$

Then there is $Q \in SO(n)$ such that

$$\|\nabla v - Q\|_{\mathrm{w}-L^1(\Omega)} \le c\varepsilon \,, \qquad \|\nabla v - Q\|_{L^1(\Omega)} \le c\varepsilon \max\left(1, \ln\frac{1}{\varepsilon}\right) , \tag{2.34}$$

and, for some $b \in \mathbb{R}^n$,

$$\|v - Qx - b\|_{L^{1^*}(\Omega)} \le c\varepsilon \,, \qquad \|v - Qx - b\|_{L^1(\partial\Omega)} \le c\varepsilon \,, \tag{2.35}$$

where $1^* = n/(n-1)$. All constants depend only on Ω.

Both results in (2.35) were first derived by Kohn in [27] for bilipschitz maps using a different measure of strain (which is equivalent to the present one after the truncation step).

Proof. By Proposition A.1 of [21] we can truncate v to obtain u such that

$$|\nabla u| \le c \,, \qquad \|\nabla u - \nabla v\|_{L^1(\Omega)} \le c\varepsilon \,.$$

It is clearly sufficient to prove the result for $\varepsilon < 1/2$, and for u instead of v. By (2.31) with $p = 2$ there is $R \in SO(n)$ such that

$$\|\nabla u - R\|_{L^2(\Omega)} \le c\|\mathrm{dist}(\nabla u, SO(n))\|_{L^2(\Omega)} \le c\varepsilon^{1/2} . \tag{2.36}$$

We can assume without loss of generality that $R = \mathrm{Id}$ (otherwise, we replace u by $R^{-1}u$). By Taylor expansion one gets

$$\left|\frac{\nabla u + \nabla u^T}{2} - \mathrm{Id}\right| \le c\,\mathrm{dist}(\nabla u, SO(n)) + c|\nabla u - \mathrm{Id}|^2$$

pointwise, which gives

$$\|\nabla u + \nabla u^T - 2\mathrm{Id}\|_{L^1(\Omega)} \le c\varepsilon \,.$$

By (2.33) there is an antisymmetric matrix $S \in \mathbb{R}^{n\times n}$ such that

$$\|\nabla u - \mathrm{Id} - S\|_{\mathrm{w}-L^1(\Omega)} \le c \left\|\nabla u + \nabla u^T - 2\mathrm{Id}\right\|_{L^1(\Omega)} \le c\varepsilon \,,$$

and by (2.36) one has $|S| \le c\varepsilon^{1/2}$. By Taylor expansion one finally obtains

$$\|\nabla u - e^{-S}\|_{\mathrm{w}-L^1(\Omega)} \le c\varepsilon \,.$$

This proves the weak-L^1 estimate in (2.34). The trace estimate and the L^{1^*} follow analogously from the corresponding linear results. To prove the L^1 estimate for ∇u, we simply observe that

$$\begin{aligned} \|\nabla u - Q\|_{L^1(\Omega)} &= \int_0^\infty |\{x \in \Omega : |\nabla u(x) - Q| > t\}| dt \\ &\leq c\|\mathrm{dist}(\nabla u, SO(n))\|_{L^1(\Omega)} + \int_0^{4n} |\{x : |\nabla u - Q| > t\}| dt \\ &\leq c\varepsilon + \int_0^{4n} \min\left(|\Omega|, \frac{1}{t}\|\nabla u - Q\|_{\mathrm{w}-L^1}\right) dt \leq c\varepsilon \ln \frac{1}{\varepsilon}\,. \end{aligned}$$

Optimality of this result was shown in [11]. □

In closing, we remark that the proof of Friesecke, James and Müller holds for all $p \in (1, \infty)$ on smooth domains, just replacing (3.13) in [21] with the corresponding singular-integral estimate for the Laplace operator. However, L^p estimates for the inhomogeneous Dirichlet problem do not, in general, hold on Lipschitz domains, see [23]. This difficulty can be circumvented by solving the Laplace equation on the whole of $\mathbb{R}^n$. Precisely, one replaces (3.12) in [21] by $z = \mathrm{div}\,\psi$, where $\psi : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ solves

$$-\Delta\psi = \begin{cases} \mathrm{cof}\,\nabla v - \nabla v & \text{in } \Omega \\ 0 & \text{outside}\,, \end{cases}$$

componentwise, with zero boundary data at infinity. Then, singular-integral estimates on $\mathbb{R}^n$ give

$$\|\nabla z\|_{L^p(\Omega)} \leq \|\nabla^2\psi\|_{L^p(\Omega)} \leq \|\nabla^2\psi\|_{L^p(\mathbb{R}^n)} \leq c\|\,\mathrm{cof}\,\nabla v - \nabla v\|_{L^p(\Omega)}\,,$$

which replaces (3.13). The same applies to Theorem 3.1. The rest of the argument is unchanged.

3 Gamma convergence

In this section we determine the Gamma limit of

$$I_\varepsilon[u,\Omega] = \int_\Omega \frac{1}{\varepsilon} W(\nabla u) + \varepsilon |\nabla^2 u|^2 dx\,.$$

as $\varepsilon \to 0$. We always assume that

$$\begin{aligned} &W \in C^0(\mathbb{R}^{2\times 2},\mathbb{R}) \text{ satisfies } W(QF) = W(F) \text{ for all } F \in \mathbb{R}^{2\times 2}, Q \in SO(2),\\ &W \geq 0,\ W \text{ vanishes on } K = SO(2)\{A,B\} \text{ with } \det A > 0\,,\ \det B > 0\,. \end{aligned} \tag{3.1}$$

We additionally assume quadratic growth, i.e. for constants $c_1, c_2 > 0$

$$c_1 \operatorname{dist}^2(F,K) \leq W(F) \leq c_2 \operatorname{dist}^2(F,K). \tag{3.2}$$

We show that the limit functional is finite only on functions u such that ∇u takes only values in K, and that on such functions it is proportional to the length of the interface between the region where $\nabla u \in SO(2)A$ and the one where $\nabla u \in SO(2)B$. Dolzmann and Müller [16] have characterized such functions as local laminates that are locally affine and have jumps only between the A and the B region. The Gamma limit of I_ε is

$$I_0[u,\Omega] = \begin{cases} \int_{J_{\nabla u}} k(\nu) d\mathcal{H}^1 & \text{if } \nabla u \in BV(\Omega, K),\\ +\infty & \text{else,} \end{cases} \tag{3.3}$$

where $J_{\nabla u}$ denotes the jump set of ∇u and ν the normal to it. The surface energy k is defined as

$$k(\nu) = \inf\left\{ \liminf_{i\to\infty} I_{\varepsilon_i}[u_i, Q_\nu] : \varepsilon_i \to 0, u_i \to u_0^\nu \text{ in } L^1 \right\}\,, \tag{3.4}$$

is positive, and satisfies $k(\nu) = k(-\nu)$. Here, Q_ν is a unit square centered in the origin with one side parallel to ν and u_0^ν is a continuous function with $\nabla u_0^\nu(x) = A$ if $x \cdot \nu > 0$, and $\nabla u_0^\nu(x) = QB$ if $x \cdot \nu < 0$, $Q \in SO(2)$ being such that $A - QB = a \otimes \nu$ for some $a \in \mathbb{R}^2$.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a strictly star-shaped, bounded Lipschitz domain and let W satisfy (3.1) and (3.2). Then*

$$\Gamma - \lim_{\varepsilon\to 0} I_\varepsilon = I_0$$

with respect to the strong L^1 topology, and finite-energy sequences have a converging subsequence. More precisely, *(i).* **Compactness.** *For every sequence u_i, ε_i with $\varepsilon_i \to 0$ and $I_{\varepsilon_i}[u_i, \Omega] \le C$ there exists a subsequence, again denoted by ε_i, u_i, and u_0 with $I_0(u_0) < \infty$ such that $u_i - p_i \to u_0$ in $W^{1,2}(\Omega)$ for some $p_i \in \mathbb{R}$.*

(ii). **Lower bound.** *For every function u_0 and every sequences $\varepsilon_i \to 0$ and $u_i \to u_0$ in $L^1(\Omega)$ we have*

$$\liminf_{i\to\infty} I_{\varepsilon_i}[u_i, \Omega] \ge I_0(u_0, \Omega).$$

(iii). **Upper bound.** *For every function u_0 and every sequence $\varepsilon_i \to 0$ there exists a sequence $u_i \to u_0$ in $L^1(\Omega)$ such that*

$$\lim_{i\to\infty} I_{\varepsilon_i}[u_i, \Omega] \le I_0(u_0, \Omega).$$

An open set Ω is strictly star-shaped if there is a point $x \in \Omega$ such that for any $y \in \partial\Omega$ the segment (x, y) is contained in Ω. The theorem is proven in the remaining subsections. The argument can be extended to finitely many wells with minor changes.

3.1 Compactness

We start with the compactness result. The standard argument, based on the arithmetic-geometric mean inequality, only gives a bound for the length of the interface between the two wells. The rigidity inside each well is then obtained using the structure of $SO(n)$. We formulate the result in n dimensions since there is no change in the proof.

Proposition 3.2 (Compactness). *Assume that $W : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is continuous, with $W = 0$ on $K = SO(n)\{A, B\}$ and $W > 0$ elsewhere, for some nonsingular A and B, and*

$$W(\xi) \ge c\,|\xi|^p - \frac{1}{c}$$

for some $p \ge 1$ and all $\xi \in \mathbb{R}^{n\times n}$, and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then for all sequences u_i, ε_i such that $\varepsilon_i \to 0$ and $I_{\varepsilon_i}[u_i, \Omega] \le C < \infty$ there exists a subsequence such that

$$u_i - \frac{1}{|\Omega|}\int_\Omega u_i(x)dx$$

converges strongly in $W^{1,p}$ to a $u_0 \in W^{1,p}(\Omega)$ with $\nabla u_0 \in BV(\Omega, K)$. Further, u_0 is locally a simple laminate, in the sense that the essential boundary *of the set $E = \{x : \nabla u(x) \in SO(n)B\}$ consists of subsets of hyperplanes which extend up to the boundary of Ω and do not intersect in Ω, and u_0 is affine on each ball whose intersection with ∂E has zero $\mathcal{H}^{n-1}$-dimensional measure.*

Proof. To simplify the notation, without loss of generality we assume that $\int_\Omega u_i(x)dx = 0$ for all i. By the growth assumption on W there are L and $c > 0$ such that

$$cW(\xi) \geq |\xi|^p \qquad \text{for all } |\xi| \geq L\,.$$

Then,

$$0 \leq \int_{\{|\nabla u_i| \geq L\}} |\nabla u_i|^p dx \leq c \int_\Omega W(\nabla u_i)dx \leq cC\varepsilon_i \to 0\,. \tag{3.5}$$

Therefore the sequence u_i is equibounded in $W^{1,p}$, and for $p = 1$ the sequence ∇u_i is equiintegrable. It follows that we can extract a subsequence that converges weakly in $W^{1,p}$ to a limit, call it u_0. At the same time, ∇u_i generates a gradient Young measure $\{\nu_x\}_{x\in\Omega}$, and

$$0 = \lim_{i\to\infty} \int_\Omega W(\nabla u_i)dx \geq \int_\Omega \int_{\mathbb{R}^{n\times n}} W(\xi)d\nu_x(\xi)dx\,.$$

Therefore the measure ν_x is supported on K for almost every $x \in \Omega$.

Consider now the truncated geodesic distance $d_W(F, G)$ induced on $\mathbb{R}^{n\times n}$ by the potential W, which is defined by

$$\begin{aligned} d_W(F,G) \quad = \quad & \inf \left\{ \int_0^1 \min\left(\sqrt{W(g(s))}, L\right) |g'(s)|ds : g \in C^0([0,1], \mathbb{R}^{n\times n}), \right. \\ & \left. \qquad\qquad g(0) = F, g(1) = G, g \text{ piecewise } C^1 \right\}\,, \end{aligned} \tag{3.6}$$

where L is as above. The function $d_W(\cdot, A)$ is Lipschitz continuous, and $d_W(F, A) = 0$ iff $F = QA$ for some $Q \in SO(n)$, and the same for B. If $AB^{-1} \notin SO(n)$, since W is positive away from K we get $d_W(B, A) > 0$.

From the definition we have

$$\int_\Omega |\nabla d_W(\nabla u_i(x), A)| \leq \int_\Omega \sqrt{W(\nabla u_i(x))}\,\left|\nabla^2 u_i(x)\right| dx \leq \frac{1}{2} I_{\varepsilon_i}[u_i, \Omega]\,.$$

Therefore $d_W(\nabla u_i(x), A)$ is uniformly bounded in $W^{1,1}$, and taking an additional subsequence, we have that $d_W(\nabla u_i(x), A)$ converges weakly in BV to a limit $d_0(x) = \int d_W(\xi, A)d\nu_x(\xi)$. If $AB^{-1} \notin SO(n)$ the limit d_0 takes only two values, 0 and $d_W(B, A) > 0$, and

$$d_W(\nabla u_i(x), A) \rightharpoonup d_0(x) = d_W(B, A)\chi_E(x) \qquad \text{weakly in BV}$$

for some set of bounded perimeter $E \subset \Omega$. In the degenerate case $AB^{-1} \in SO(n)$ it suffices to take $E = \emptyset$ in the following argument. In both cases we have $\operatorname{supp} \nu_x \in SO(n)B$ a.e. on E, and $\operatorname{supp} \nu_x \in SO(n)A$ a.e. on $\Omega \setminus E$.

In order to show that $u_i \to u_0$ strongly in $W^{1,p}$, we first need to truncate the sequence. This is done by means of Proposition A.1 of [21] (see also [17, Sect. 6.6.2]), which permits to construct for each u_i a function v_i such that

$$\|\nabla v_i\|_{L^\infty(\Omega)} \le c, \qquad \|\nabla u_i - \nabla v_i\|_{L^1(\Omega)} \le c \int_{\{|\nabla u_i| > L\}} |\nabla u_i| dx \le c\varepsilon_i \,.$$

Therefore v_i has the same weak limit (in $W^{1,p}$) as u_i, and its gradients generate the same Young measure ν_x. Since the sequence ∇v_i is uniformly Lipschitz, the limit formula can be applied to all continuous functions.

The proof is then concluded using a rigidity result for $W^{1,n}$ functions which was first derived by Reshetnyak [35]. Instead of using the result, we incorporate here the simple proof of his statement given by Kinderlehrer [25] and Müller [32]. For a fixed representative ∇u_0 let Ω_0 be the set of Lebesgue points, i.e. the set of points $x_0 \in \Omega$ such that

$$\lim_{r\to 0} \frac{1}{|B_r|} \int_{B_r} |\nabla u_0(x) - \nabla u_0(x_0)| \, dx = 0 \,. \tag{3.7}$$

Furthermore, let Ω_A be the set of points where E has vanishing density, i.e. of points $x_0 \in \Omega$ such that

$$\lim_{r\to 0} \frac{|B(x_0, r) \cap E|}{|B(x_0, r)|} = 0 \,.$$

We claim that

$$x_0 \in \Omega_0 \cap \Omega_A \qquad \text{implies} \qquad \nabla u_0(x_0) \in SO(n)A \,.$$

To see this we fix a sequence $r_j \to 0$ and consider the sequence of balls $B_j = B(x_0, r_j)$. We study the polyconvex function $\varphi : \mathbb{R}^{n\times n} \to \mathbb{R}$ defined by

$$\varphi(\xi) = |\xi|^n - n^{n/2} \det \xi$$

(with $|\xi|^2 = \operatorname{Tr} \xi^T \xi$), which is nonnegative and vanishes only on matrices which are scalar multiples of matrices in $SO(n)$ (by isotropy it is enough to consider diagonal matrices, for which this property follows from the arithmetic–geometric mean inequality). Then for a fixed j we have

$$\lim_{i\to\infty} \int_{B_j} \varphi(\nabla v_i A^{-1}) dx = \int_{B_j} \int_{\mathbb{R}^{n\times n}} \varphi(\xi A^{-1}) d\nu_x(\xi) dx \,.$$

The measure ν_x is supported on K, and the function $\varphi(\xi A^{-1})$ takes only the two values 0 and $\varphi(BA^{-1})$ on it. Therefore

$$\begin{aligned} \lim_{i\to\infty} \int_{B_j} \varphi(\nabla v_i A^{-1}) dx &= \varphi(BA^{-1}) \int_{B_j} \nu_x(SO(n)B) dx \\ &= \varphi(BA^{-1}) |B_j \cap E| \,. \end{aligned} \tag{3.8}$$

The function φ is polyconvex, and therefore

$$\int_{B_j} \varphi(\nabla u_0 A^{-1}) dx \le \liminf_{i\to\infty} \int_{B_j} \varphi(\nabla v_i A^{-1}) dx = \varphi(BA^{-1}) |B_j \cap E| \,.$$

We conclude

$$\lim_{j\to\infty} \frac{1}{|B_j|} \int_{B_j} \varphi(\nabla u_0 A^{-1}) dx = 0 \,.$$

But now (3.7) yields $\varphi(\nabla u_0(x_0)\, A^{-1}) = 0$. The same argument replacing the function φ with det and $-\det$ shows that $\det(\nabla u_0(x_0) A^{-1}) = 1$. But

$$\varphi(\xi) = 0 \text{ and } \det \xi = 1 \qquad \text{implies} \qquad \xi \in SO(n),$$

therefore $\nabla u_0(x_0) \in SO(n)A$ and the claim is proven. Reasoning the same way in $\Omega \setminus E$ we obtain that $\nabla u_0 \in BV(\Omega, K)$. Then, the conclusion follows from the characterization of functions u_0 with $\nabla u_0 \in BV(\Omega, K)$ given by Dolzmann and Müller [16, Theorem 1.1].

We finally prove strong convergence of ∇v_i. Let $F_i = \nabla v_i (\nabla u_0)^{-1}$. Since $\nabla v_i \rightharpoonup \nabla u_0$ we have $F_i \rightharpoonup \mathrm{Id}$ in L^p, hence to prove strong convergence it suffices to show that $|F_i| \to |\mathrm{Id}|$ in L^p. We compute

$$\lim_{i\to\infty} \int_\Omega |F_i|^p dx = \int_\Omega \int_{\mathbb{R}^{n\times n}} |\xi \nabla u_0^{-1}(x)|^p d\nu_x(\xi) dx = |\Omega|\, |\mathrm{Id}|^p \,,$$

since $\operatorname{supp} \nu_x \in SO(n) \nabla u_0(x)$. This concludes the proof. ☐

3.2 Lower bound

This subsection is devoted to point (ii) of Theorem 3.1.

Proposition 3.3 (Lower bound). *Let W satisfy (3.1) and let Ω be a bounded, Lipschitz domain. Then, for all sequences $\varepsilon_i \to 0$ and $u_i \to u_0$ in L^1, we have*

$$\liminf_{i\to\infty} I_{\varepsilon_i}[u_i, \Omega] \ge I_0[u_0, \Omega] \,. \tag{3.9}$$

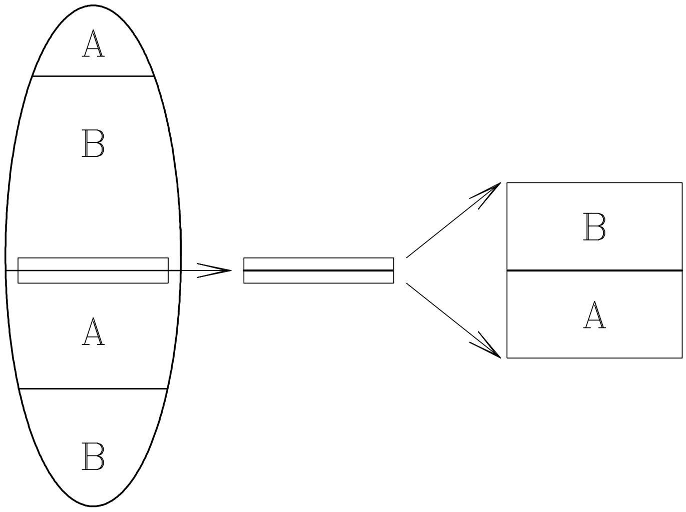

FIGURE 3.1: To prove the lower bound we first isolate rectangles around the interfaces and then show that the energy per unit interfacial length does not depend on the shape or size of the rectangle.

Sketch of proof. The argument is completely analogous to the one in [12, 13]. We only sketch the main ideas, which are illustrated in Figure 3.1.

From general u_0 to a single interface in a rectangle. By the compactness result, we only have to consider limiting functions u_0 which are locally laminates. For any $\eta > 0$, one can cover a fraction $1-\eta$ of the total interfacial length by finitely many rectangles, each of them containing a single interface in the middle. Since η is arbitrary, and the total energy is larger than or equal to the sum of the energy contained in the rectangles, it is sufficient to prove the result for each such rectangle.

The rectangles come in four variants: the interface can have two orientations, and A and B can be located on each side of the interface. We identify them by the normal ν, which we take to be oriented from the A to the B phase. For notational simplicity we assume $\nu = e_2$. Then, the optimal limiting energy on a $2d \times 2l$ rectangle is given by

$$\mathcal{F}_\nu(d,l,R) = \inf\left\{\liminf_{i\to\infty} I_{\varepsilon_i}[u_i,(-d,d)\times(-l,l)] : \varepsilon_i \to 0, u_i \to u_0^\nu \text{ in } L^1\right\},$$

where for $R \in SO(2)$

$$u_0^\nu(x) = \begin{cases} RAx & \text{if } x_2 < 0 \\ RQBx & \text{if } x_2 \geq 0, \end{cases}$$

Q being the unique rotation such that $A - QB = a \otimes e_2$ for some $a \in \mathbb{R}^2$.

Identification of relevant parameters. By the rotational invariance of W the number $\mathcal{F}_\nu(d,l,R)$ is independent of R. Replacing $u(x)$ by $-u(-x)$ we find that $\mathcal{F}_\nu = \mathcal{F}_{-\nu}$. The scaling and covering argument of [12] shows that $\mathcal{F}_\nu(d,l,R)$ is independent of l and depends linearly on d. It is therefore sufficient to evaluate it on a unit square and we find $\mathcal{F}_\nu(d,l,R) = 2dk(\nu)$, where k was defined in (3.4). ☐

The proof of the lower bound is completely abstract, and based only on the known structure of the limiting function. The argument does not bring any control on low-energy sequences, which has to be obtained separately by means of a rigidity estimate. This is the topic of the next section.

3.3 $H^{1/2}$ rigidity

The main difficulty in proving Theorem 3.1 is to obtain the upper bound in item (iii). The principal idea is to construct approximating sequences from the energy-minimizing sequences of (3.4). If multiple interfaces are present, it is necessary to interpolate between the minimizing sequences corresponding to each interface. A matching with low energy is possible if the deformation is sufficiently close to an affine function on a horizontal line, in $H^{1/2}$. Building upon the segment rigidity of Proposition 2.2, we shall now derive an optimal rigidity estimate in $H^{1/2}$ on lines, which will then be used in the next section to obtain the upper bound.

The function ϕ appearing in the proposition will later be set to be the distance d_W induced by W, as defined in (3.6). In this subsection we use ξ and ζ to denote horizontal and vertical coordinates, $x = (\xi, \zeta) \in \mathbb{R}^2$.

Proposition 3.4. *Let $\Omega = (-d,d) \times (-l,l)$ be a rectangle in $\mathbb{R}^2$, A and B have positive determinant, and $\phi \in C^1(\mathbb{R}^{2\times 2}, \mathbb{R})$ be a function satisfying $\phi(F) \geq \bar{c}\,\mathrm{dist}^p(F, SO(2)A)$ for some $\bar{c} > 0$, $p \geq 1$. Then there are constants $c, \eta > 0$ such that for every function $u : \Omega \to \mathbb{R}^2$ with*

$$\int_\Omega \phi(\nabla u) + |\nabla[\phi(\nabla u)]| \leq \eta \tag{3.10}$$

there is a subset $\mathcal{Y} \subset (-l,l)$ of measure at least $3l/2$ such that for any $\zeta \in \mathcal{Y}$,

$$\min_{Q\in SO(2),\, b\in\mathbb{R}^2} \|u - QAx - b\|^2_{H^{1/2}(\Gamma_0)} \leq c \int_\Omega \mathrm{dist}^2(\nabla u, SO(2)\{A,B\})dx\,, \tag{3.11}$$

where $\Gamma_0 = (-d,d) \times \{\zeta_0\}$.

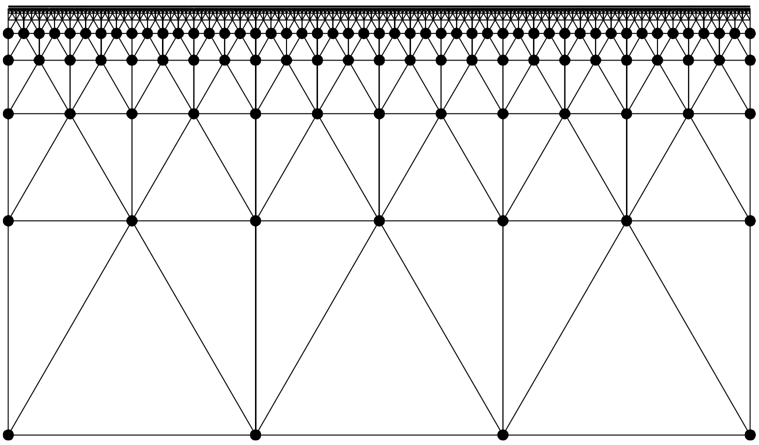

FIGURE 3.2: The reference grid. Here $K = 3$, hence the largest triangles have side $l_0 = 2d/3$, height $h_0 = l_0\sqrt{3}/2$, and are located in the strip $-2h_0 \leq \zeta - \zeta_0 \leq h_0 = 2h_1$.

Proof. The proof is based on the segment rigidity of Proposition 2.2. The main idea is to construct a function v which has small one-well energy and the same trace as u on Γ_0. This is done by defining v as the piecewise affine interpolation of the values of u on the vertices of a suitable grid. The grid is constructed in such a way that all grid edges are approximately rigid in the sense of Proposition 2.2, which implies that the distance of ∇v from $SO(2)A$ is controlled by the distance of ∇u from K, in an L^2 sense (see below). Then, by the one-well Friesecke-James-Müller rigidity stated in (2.31) we obtain an equivalent control of the distance of ∇v from a constant, and the trace theorem gives the estimate for v in $H^{1/2}(\Gamma_0)$. Since the grid can be chosen so that it refines towards Γ_0, we get $u = v$ on Γ_0 and the estimate for u.

We now start the proof. By a change of variables we can assume $A = \mathrm{Id}$, and we focus on the case $\zeta_0 > 0$. For a large subset of ζ_0's in $(0, l)$ we have (see [13, Step 2 of Lemma 4.5])

$$\frac{1}{\delta} \int_{(-d,d)\times(\zeta_0-\delta,\zeta_0)} \phi + |\nabla\phi| \leq c\eta \tag{3.12}$$

for all $\delta \in (0, l)$.

Construction of the reference grid. We construct a triangular grid as indicated in Figure 3.2, which is based on equilateral triangles and refines towards the line $\Gamma_0 = (-d, d) \times \{\zeta_0\}$. Precisely, let K be the integer part of $1 + d/l$, and set $l_k = 2d\,2^{-k}/K$, $h_k = l_k\sqrt{3}/2$. On each line $\zeta = \zeta_0 - 2h_k$ we fix $K2^k + 1$ equispaced vertices (the spacing being l_k), and join them as

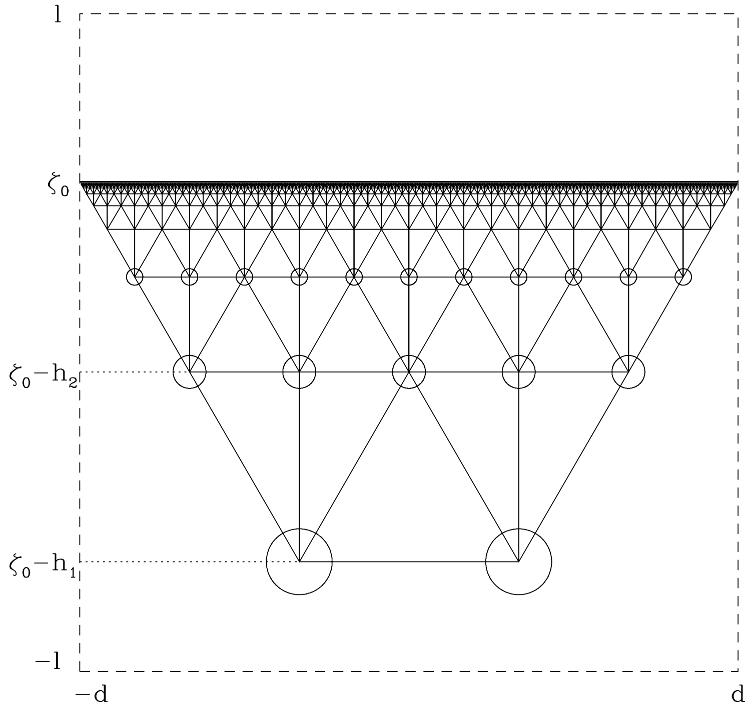

FIGURE 3.3: The inner part of the reference grid. The dashed rectangle represents Ω, the circles represent the balls B_m.

illustrated in the figure to form triangles which are either equilateral or have angles of 30, 60 and 90 degrees. The grid covers $(-d,d)\times(\zeta_0-\min(l,d)/2,\zeta_0)$, is contained in Ω, and the length of each edge is controlled from above and below by its distance h_k from Γ_0.

For later use we now show that (3.12) implies a local control of ϕ in L^1 with the optimal scaling. We consider an interval $Y_k = (\zeta_0 - h_k, \zeta_0)$ and rewrite (3.12) as

$$\int_{Y_k} e(\zeta)\, d\zeta \le c\eta h_k\,, \quad \text{where} \quad e(\zeta) = \int_{-d}^{d} \phi(\xi,\zeta) + |\nabla\phi(\xi,\zeta)|\, d\xi\,.$$

Hence there is $\zeta_k \in Y_k$ such that $e(\zeta_k) \le c\eta$. The one-dimensional embedding $W^{1,1} \subset L^\infty$ yields that $\phi(\cdot,\zeta_k)$ is bounded by $c'\eta$ pointwise on $(-d,d)$. The Poincaré inequality for the square $S = (\xi - h_k, \xi) \times (\zeta_0 - h_k, \zeta_0)$ yields, for arbitrary $\xi \in (-d + h_k, d)$,

$$\int_S \phi\, dx \le h_k^2 \|\phi(\cdot,\zeta_k)\|_{L^\infty(-d,d)} + h_k \int_S |\nabla\phi|\, dx \le c\eta h_k^2. \tag{3.13}$$

Construction of the perturbed grid. Since the segment-rigidity of Proposition 2.2 is an interior estimate, we restrict to the triangles whose closure is entirely contained in the open set Ω, as indicated in Figure 3.3. We consider balls $B_m = B(v_m, r_m)$ centered in the grid points $(v_m)_{m\in\mathbb{N}}$, where the radius r_m is $l_k/10$ for vertices of level k, and apply Proposition 2.2 to each pair of neighboring vertices. Let $n = (m, m')$ denote one pair of neighboring vertices at level k. Then by (3.12,3.13) we can apply Proposition 2.2 to it, provided that η is chosen appropriately. Hence there are many pairs $(w_m, w_{m'}) \in B_m \times B_{m'}$ such that the segment $[w_m, w_{m'}]$ is rigid.

We now choose inductively one point w_m in each B_m. We start by saying that all points of all balls are *possible choices in step* 0. At step m we set w_m as one of the *possible choices in step* m in the ball B_m with the additional condition that w_m forms a rigid pair with many points of all neighboring balls $B_{m'}$ with $m' > m$. This is possible provided that in B_m there are many possible choices in step m, since there is only a finite number of neighbors. The set of *possible choices in step* $m+1$ then consists of the possible choices in step m without those $w \in B_{m'}$ such that $m' > m$ is a neighbor of m, but (w, w_m) is not a rigid pair. Since each point has at most seven neighbors, we reduce the possible choices of each ball B_m finitely many times by a small set, whence at all steps there remain many possible choices in each ball where the choice has not been made yet.

The piecewise linear interpolation. For every edge $e_n = [x, y]$ of level k, Proposition 2.2 yields

$$\left|\frac{|u(x)-u(y)|}{|x-y|} - 1\right| \quad\leq\quad c\frac{1}{l_k^2}\|\mathrm{dist}(\nabla u, K)\|_{L^1(\omega_n)} \leq c\frac{1}{l_k}\|\mathrm{dist}(\nabla u, K)\|_{L^2(\omega_n)}.$$

Here ω_n is a ball containing the edge e_n, whose radius is less than the length of e_n. Each point in the domain is contained in at most finitely many such balls.

We conclude that for every triangle T of the perturbed grid, the linear interpolation $v : T \to \mathbb{R}^2$ between the values of u on the vertices is close to a rigid motion. By (3.13) we get $\det \nabla v > 0$, hence

$$\|\mathrm{dist}(\nabla v, SO(2))\|_{L^2(T)} \leq c\sum_{i=1}^{3} \|\mathrm{dist}(\nabla u, K)\|_{L^2(\omega_{n_i})}.$$

We take squares and sum over all triangles of level k to find

$$\int_{(-d,d)\times(\zeta_0-l_k,\zeta_0-l_{k+1})} \mathrm{dist}^2(\nabla v, SO(2))\, dx \\ \leq c\int_{(-d,d)\times(\zeta_0-2l_k,\zeta_0-\frac{1}{4}l_k)} \mathrm{dist}^2(\nabla u, K)\, dx\,.$$

Summing over k yields

$$\int_\Omega \mathrm{dist}^2(\nabla v, SO(2))dx \le c\int_\Omega \mathrm{dist}^2(\nabla u, SO(2)\{A,B\})\,dx\,. \tag{3.14}$$

By the Friesecke-James-Müller rigidity (2.31) there is a rotation $Q \in SO(2)$ such that the left-hand side provides a bound for the distance to Q,

$$\int_\Omega |\nabla v - Q|^2 dx \le c\int_\Omega \mathrm{dist}^2(\nabla u, SO(2)\{A,B\})\,dx\,. \tag{3.15}$$

The thesis follows by the trace theorem. □

3.4 Upper bound

We now prove the upper bound for Theorem 3.1. The construction is essentially the same as in the geometrically linear case [13]; for the convenience of the reader we sketch the main ideas and refer to [13] for details.

Proposition 3.5. *Let W satisfy (3.1) and (3.2), $\Omega \subset \mathbb{R}^2$ be open, bounded, and strictly star-shaped. Then, for every function u_0 and every sequence $\varepsilon_i \to 0$ there exists a sequence $u_i \to u_0$ in $L^1(\Omega)$ such that*

$$\lim_{i\to\infty} I_{\varepsilon_i}[u_i,\Omega] \le I_0[u_0,\Omega]\,.$$

Sketch of proof. We consider only the case $I_0(u_0,\Omega) < \infty$ since the other case is trivial. Thus we are given $u_0 \in L^1(\Omega)$ with $\nabla u_0 \in BV(\Omega, K)$, i.e. u_0 is locally a laminate.

From many interfaces to one interface. We have to construct a sequence $u_i \to u_0$ such that $I_{\varepsilon_i}[u_i] \to I_0[u_0]$. By a density argument which exploits star-shapedness it is sufficient to consider limiting functions which have finitely many interfaces which do not meet at the boundary.

We consider disjoint rectangular boxes, one covering each interface. We claim that in each such box, which after translation and assuming for notational simplicity that the interface is parallel to e_1 has the form $R = (-d,d)\times(-l,l)$, we can find w_i converging to u_0 such that w_i is affine away from the interface and has the optimal limiting energy, i.e.

$$I_{\varepsilon_i}[w_i,R] \to 2dk(e_2)\,,\;\; w_i = I_i \circ u_0 \text{ for } \zeta \ge \frac{4}{5}l\,,\;\; w_i = I_i' \circ u_0 \text{ for } \zeta \le \frac{4}{5}l\,, \tag{3.16}$$

where I_i and I_i' are isometries which converge to the identity as $i \to \infty$. Then, it is sufficient to join these functions by affine functions with gradient in K (which have zero energy) and to compose each of them by a rigid motion close to the identity. To see this, we first observe that if J_i is any sequence of isometries converging to the identity, we can assume that (3.16) holds with $I'_i = J_i$ (it suffices to compose w_i with J_i times the inverse of I'_i). Now, consider for concreteness the case of finitely many parallel interfaces in a convex domain, and choose a unique l smaller than half the distance between them. We sort the interfaces in order of ascending x_2. Below the first interface (i.e. at smaller x_2) we set $u_i = u_0$. Around the first interface we use (3.16) with I'_i set equal to the identity, and define I_i^1 as the corresponding isometry I_i on the upper side of the interface. Then we set $u_i = I_i^1 \circ u_0$ between the first and the second interface, and around the second interface we use again (3.16), now with $I'_i = I_i^1$. We then let I_i^2 be the corresponding isometry on the upper side of the interface, and continue. This procedure can be completed since the domain is simply connected, and all isometries converge to the identity since there are finitely many interfaces. Further, no discontinuity in u or ∇u is inserted in the procedure, and all affine pieces have gradient in K, hence the total energy converges to the sum of the energies of the single interfaces. Therefore it only remains to prove (3.16).

One interface: proof of (3.16). Consider one box $R = (-d, d) \times (-l, l)$ which contains a single interface in the center. By the definition of $k(\nu)$ and the compactness result there are sequences ε_i and u_i such that, as $i \to \infty$,

$$I_{\varepsilon_i}[u_i, R] \to 2dk(\nu) \qquad \text{and} \qquad \int_R |\nabla u_i - \nabla u_0|^2 \, dx \to 0\,.$$

Since the limiting energy does not depend on l, it is clear that the energy is concentrated in the smaller box $(-d, d) \times (-l/2, l/2)$. Therefore in the strip $\omega = (-d, d) \times (l/2, l)$ the energy converges to zero, and in particular

$$\int_\omega \frac{1}{\varepsilon_i} W(\nabla u_i) + |\nabla d_W(\nabla u_i, A)| \le 2I_{\varepsilon_i}[u_i, \omega] \to 0 \tag{3.17}$$

where d_W was defined in (3.6). By Proposition 3.4 applied on the domain ω with $\phi(\cdot) = d_W(\cdot, A)$, for each i there are many $\zeta_i \in (l/2, 3l/4)$ for which

$$\frac{1}{\varepsilon_i} \|u_i - Z_i\|^2_{H^{1/2}(\Gamma_i)} \to 0\,, \qquad \nabla Z_i \in SO(2)B\,,$$

for some affine Z_i. Here and below $\Gamma_i = (-d, d) \times \{\zeta_i\}$. Further, we can choose ζ_i such that

$$\frac{1}{\varepsilon_i} \int_{(-d,d)\times(\zeta_i,\zeta_i+\varepsilon_i)} |\nabla u_i - \nabla u_0|^2 + \varepsilon_i |\nabla^2 u_i|^2 dx \to 0 \tag{3.18}$$

and

$$\int_{\Gamma_i} \varepsilon_i |\nabla^2 u_i|^2 + |\nabla u_i - \nabla u_0|^2 dx \to 0 \,. \tag{3.19}$$

Since ∇Z_i and ∇u_0 only differ by a rotation, this implies $\nabla Z_i \to \nabla u_0$.

The claimed (3.16) follows then from Lemma 5.5 of [13] (and repeating the argument for $\zeta < 0$). For completeness we repeat the main ideas here. In a first step one constructs functions $v_i : \omega_i = (-d, d) \times (\zeta_i, l)$ such that $v_i = Z_i$ for $\zeta > 4l/5$, $v_i = u_i$ on $\zeta = \zeta_i$, and $I_\varepsilon[v_i, \omega_i] \to 0$. In this step one controls the nonconvex part of the energy by the trace theorem, i.e.

$$\int_{\omega_i} \frac{1}{\varepsilon_i} W(\nabla v_i) dx \le c \int_{\omega_i} \frac{1}{\varepsilon_i} |\nabla v_i - \nabla Z_i|^2 dx \le c \|u_i - Z_i\|^2_{H^{1/2}(\Gamma_i)} \to 0 \,.$$

The expression $|\nabla v_i - \nabla Z_i|^2$ is convex, hence is not increased by mollification. By the first term in (3.19) one can mollify so that the boundary values on Γ_i are unchanged and the singular perturbation $\int_{\omega_i} \varepsilon_i |\nabla^2 v_i|^2 dx$ becomes infinitesimal. The precise construction can be done either by a suitable mollification of the piecewise affine construction used in the proof of the $H^{1/2}$ estimate, or by harmonic extension, or explicitly in Fourier space, as explained in Lemma 5.4 of [13]. In a second step one defines w_i as an interpolation between u_i and v_i on the strip $\zeta \in (\zeta_i, \zeta_i + \varepsilon_i)$, and exploits (3.18) to control the error terms. The argument is finally repeated for negative ζ, hence (3.16) follows (for the same sequence $\varepsilon_i \to 0$ chosen at the beginning of this step).

It remains to show that one can construct u_i for *any* sequence $\varepsilon_i \to 0$. This is done by means of an upscaling and compactness argument for which we refer to Proposition 5.6 of [13]. $\square$

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