



**Workshop:
Nonlinear and Adaptive Approximation in
High Dimensions**

Physikzentrum Bad Honnef, Germany
December 10-15, 2007

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Invited Speakers

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Albert Cohen (Paris VI)
Ron DeVore (University of South Carolina)
Thomas Gerstner (Universität Bonn)
Helmut Harbrecht (Universität Bonn)
Claude Le Bris (ENPC-INRIA)
Christian Lubich (Universität Tübingen)
Mauro Maggioni (Duke University)
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Abstracts

Sparse Occupancy Trees and Adaptive Approximation in High Dimensions

PETER G. BINEV

(joint work with Wolfgang Dahmen and Ron DeVore)

We consider the problem of finding an adaptive approximation to a function $f : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$ which is known to us via certain observation information, e.g. point evaluations $y^i = f(x^i)$ at a set of points $X = \{x^i\}_i \subset \mathbb{R}^d$ or more generally a set of evaluations of certain functionals. The number N of the observations is relatively small compared to 2^d . It is often the case that there is no a priori information about the distribution of the data or its general behavior. It is unrealistic to try to find a general method that works in all kind of possible situations. Usually, the coordinates of the points from X are interdependent and the domain $D \in \mathbb{R}^d$ for f is (close to) a low-dimensional set. That constitutes an important class of problems in which one can transform the domain D into a lower dimensional one.

An additional requirement to the algorithm is to be able to process streaming data with the same complexity as the static data. That rules out most of the currently available techniques (e.g. the ones based on an embedding of \mathbb{R}^d into an Euclidean space of dimension $C_0 \log N$ given by the Johnson-Lindenstrauss Lemma).

Our approach is based on adaptive partition into cells and its association with a decision tree that records all the cells emerging in the process. The standard techniques from multiresolution analysis and adaptive methods for approximation are not applicable in high space dimension d since the number of nodes in the corresponding tree is exponential in d . To overcome this difficulty we introduce the notion of *sparse occupancy trees*. The idea is to relate the adaptive partitions to trees whose nodes are limited to the ones corresponding to cells occupied by the data points. A special indexing and ordering of these cells allows storing all of the information about the tree using for each data point a number of bits proportional to $d \log N$ and processing time of order $\mathcal{O}(d \log N)$. This structure can be used as a tool to build different algorithms for approximation. A straight forward one is the piecewise constant approximation which is fast and easy to implement but may not yield the accuracy needed for a given problem. Realizing higher accuracy requires better proximity information. To provide it, we consider trees based on *oldest edge bisection* procedure for which each cell is a simplex. The vertices of the occupied simplices are organized in an ordered list which helps to relate the information between the neighboring cells and estimate the appropriate local depth of the tree. In particular, this framework gives the opportunity to define a piecewise linear continuous approximation on the adaptive partition determined by the tree. The function realizing this approximation cannot be given explicitly due to the large number of cells in the partition. Instead, it is defined as an $\mathcal{O}(d \log N)$ procedure which calculates the functional value at each particular query x .

**Sparse p -version BEM
for first kind boundary integral equations
with random data**

ALEXEY CHERNOV

(joint work with Christoph Schwab)

We consider the weakly singular boundary integral equation $\mathcal{V}u = g$ on a randomly perturbed smooth closed surface $\Gamma(\omega)$ with deterministic g or on a deterministic closed surface Γ with stochastic $g(\omega)$. The aim is the computation of the moments $\mathcal{M}^k u := \mathbb{E}[\otimes_{i=1}^k u]$, $k \geq 1$, if the corresponding moments of the perturbation are known. The problem on the stochastic surface is reduced to a problem on the nominal deterministic surface Γ with the random perturbation parameter $\kappa(\omega)$. Note, that $u(\omega)$ depends nonlinearly on $\kappa(\omega)$.

Resulting formulation for the k th moment is posed in the tensor product Sobolev spaces and involve the k -fold tensor product operators $\mathcal{V}^{(k)} = \otimes_{i=1}^k \mathcal{V}$. The standard full tensor product Galerkin BEM requires $\mathcal{O}(N^k)$ unknowns for the k th moment problem, where N is the number of unknowns needed to discretize the nominal surface Γ . Based on [3], we develop the p -sparse grid Galerkin BEM to reduce the number of unknowns to $\mathcal{O}(N(\log N)^{k-1})$ (cf. [1], [2] for the wavelet approach).

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Theory and algorithms for anisotropic triangulations

ALBERT COHEN

(joint work with Jean-Marie Mirebeau)

We shall present the first results of an ongoing project revolving around approximation by finite element functions on adaptive and anisotropic triangulations. We shall first recall the available theory for isotropic triangulations which involves Besov spaces.

The results that we shall present for anisotropic triangulations are of two flavors.

On the theoretical side, we shall propose an analytic criterion that seems to govern the rate of convergence in L^p norms for optimally built triangulations. We shall also discuss how this rate can be achieved by algorithms allocating the triangles according to the local Hessian of the function to be approximated.

On a more practical side, we shall discuss how certain type of greedy algorithms have the ability to generate triangulations that are in some sense close to optimal. We shall also give an application in image representation and compression.

Sparse nonlinear approximation of analytic function in high dimension

RONALD A. DEVORE

(joint work with Christoph Schwab)

We shall discuss the approximation of analytic functions in many variables (possibly infinitely many) by an n -term polynomial. Problems of this type arise in the numerical treatment of stochastic PDEs. We show the connection between this problem and certain combinatorial problems in high space dimension. We then estimate the approximation rate as $n \rightarrow \infty$ under assumptions on the Taylor coefficients of the analytic function.

An optimal adaptive method for high-dimensional elliptic PDEs

TAMMO JAN DIKEMA

(joint work with Rob Stevenson, Christoph Schwab)

We consider the problem of finding $u \in H_0^1(0,1)^d$ such that $a(u,v) = f(v)$ for all $v \in H_0^1(0,1)^d$, where a is an elliptic bilinear form. Specifically, we try to do this for large space dimensions d . When the unknown solution u is approximated using standard isotropic approximation with piecewise polynomials of a fixed degree, we run into the so-called ‘curse of dimensionality’: the convergence rate is inversely proportional to d .

Using that $(0,1)^n$ is a tensor-product domain, the curse of dimensionality can be circumvented using a sparse tensor product approximation (e.g. [Ni]). However, this can only be expected to work when special regularity conditions are met. Already for the Poisson equation with constant (non-zero) right-hand side, this is not the case.

We use an adaptive wavelet approximation method, which reaches a convergence rate as that of the best N -term approximation, in linear complexity. For this, we use orthogonal tensor product wavelets based on the multiwavelets of Donovan et al, see [DGH].

Numerical results will be shown for experiments in high dimensions, illustrating the convergence rate.

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**Efficient Best-Approximation of Tensor-Sums
in
High Dimensions with Application**

MIKE ESPIG

When an algorithm in dimension one is extended to dimension d , in almost every case its computational cost grows to the power of d . Tensor-sums (TS) are promising objects for multi-dimensional operators. Linear algebra operations can be performed in this representation using only d times one-dimensional operations. In iterative methods with tensor-sums it is important to solve the following problem.

For a given Tensor-Sum $A = \sum_{i=1}^R \otimes_{\mu=1}^d A_{i\mu} \in \text{TS}(R)$ and $\varepsilon \in \mathbb{R}_+$, find $X^* = \sum_{i=1}^{r_\varepsilon} \otimes_{\mu=1}^d X_{i\mu}^* \in \text{TS}(r_\varepsilon)$ such that:

$$\begin{aligned} (1) \quad & \|A - X^*\| \leq \varepsilon, \\ (2) \quad & \|A - X^*\| = \min_{X \in \text{TS}_c(r_\varepsilon)} \|A - X\|. \end{aligned}$$

We will introduce a numerical approach that solves this approximation problem and present numerical results in high dimensions.

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Fast numerical algorithms for applying separable integral operators in multiple dimensions

LUCA FREDIANI

(joint work with Eirik Fossgaard, Tor Flå, and Kenneth Ruud)

We have developed and implemented a formalism for the application of separable integral operators in $d \geq 1$ dimensions using the Non-Standard (NS) form [1]. The proposed formalism is general, compact and oriented towards the practical implementation into a working code using multiwavelets [2]. For the case of Poisson and Helmholtz operators we also propose a simple scheme for the generation of an approximate separated representation of the corresponding kernels with finite arbitrary precision. Such a representation [3, 4], combined with the NS-form, allows us to build a sparse, banded representation of the operator. We have implemented a code for the application of the Poisson and Helmholtz operators on a separated Non-Standard form to a multivariate function. The implemented code computes explicitly all the 2^{2d} components of the d -dimensional operator. We finally demonstrate the performance of our implementation on some test cases.

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Sparse Grids in Computational Finance

THOMAS GERSTNER

(joint work with Markus Holtz)

Computational finance is an interdisciplinary field which uses mathematical finance, stochastic methods, numerical algorithms and computer simulations to aid practitioners in banks or insurance companies with trading, hedging and investment decisions. Its main aim is to determine as accurately as possible the financial risk that financial instruments create.

Of particular interest in computational finance is the pricing of derivative securities, whose most well-known representatives are various types of options. The price of these derivatives depends on the future development of some underlying asset or a set of assets such as stocks and/or bonds.

A fundamental result from financial derivatives pricing theory is that, under certain assumptions, the fair price of a derivative security can be represented as an expected value. If the expectation is written as an integral, its dimension is in many cases high or even infinite. In nearly all cases, the arising integrals cannot be solved analytically or can be reduced into easy computational form. Thus, numerical methods are required for their solution.

Sparse grid quadrature methods are one way to overcome the curse of dimension which typically arises during the solution of such problems. The sparse grid approach is directly applicable to derivative security pricing problems which lead to smooth integrands. For many option pricing problems, however, the corresponding integrands are typically not smooth and the convergence of the sparse grid method deteriorates strongly. As a second problem, the sparse grid method is largely, but not completely independent of the dimension of the problem. This leads to a degradation of the convergence rate when the dimension increases.

In this talk, we address these two problems of missing smoothness and dimension-dependence. To this end, we develop dimension-adaptive sparse grid quadrature methods which are able to deal with non-smooth and high-dimensional problems such as they arise in computational finance. The efficiency of these methods is illustrated with so-called performance-dependent options [GH1, GH2] and the asset-liability management of life insurance contracts [GGH1, GGH2].

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Sparse second moment analysis for elliptic PDEs on stochastic domains

H. HARBRECHT

(joint work with R. Schneider and C. Schwab)

This talk is concerned with the numerical solution of Dirichlet problems in domains $D \in \mathbb{R}^d$ with random boundary perturbations. Assuming normal perturbations with small amplitude and known mean field and two-point correlation function, we derive, using a second order shape calculus, deterministic equations for the mean field and the two-point correlation function of the random solution for the Dirichlet problem in the stochastic domain.

The two-point correlation of the random solution satisfies a boundary value problem on the tensor product domain $D \times D$. It can be approximated in sparse tensor product spaces. This yields densely populated system matrices, independently of using the finite element method in $D \times D$ or the boundary element method on $\partial D \times \partial D$.

We present and analyze algorithms to approximate the random solution's two-point correlation function in essentially $\mathcal{O}(N)$ work and memory, where N denotes the number of unknowns required for consistent discretization of the domain (in case of finite element methods) or its boundary (in case of boundary element methods). Here “essentially” means up to powers of $\log N$.

Nonlinear Approximation and Tensor Product Wavelet Bases

REINHARD HOCHMUTH

An overview about already solved, in principle solvable, unsolvable and open problems in the context of n -term approximation and restricted nonlinear approximation with tensor product wavelet bases is given. Essentially solved is e.g. the task: Give intrinsic characterizations for the approximation spaces, when the approximation error is measured in L_2 -type norms and *nice* bases are considered, cf. [CDH, H, N, V]. For *less nice* bases there seem to arise only technical difficulties. But there are severe hints, that in the case of L_p similar characterizations are not true, cf. [KP, W]. At least in general, since there are certain Haar type bases, cf. [BK], which allow such characterizations. If there are related wavelet bases seems to be an open problem.

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Sparse Approximation and Compressed Sensing

SADEGH JOKAR

(joint work with Volker Mehrmann, Marc Pfetsch and Harry Yserentant)

Often signal information can be recovered by performing relatively few measurements. The paradigm of *Compressed Sensing*[D] tries to exploit this and suggests a new way in how information is collected and processed. Specifically, consider an underdetermined system of linear equations $b = \Phi x$ with known b and an $m \times n$ matrix Φ with $m < n$. We seek the sparsest solution to this system[CDD, CRT], i.e., the x with fewest nonzeros satisfying $b = \Phi x$. In general, this problem is NP-hard. In this talk, we will review some of the recent result in this field. Then we will empirically investigate several approximative methods for this[JF] and related problems.

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Tensor approximation of Green iterations for Hartree-Fock/Kohn-Sham equations

BORIS KHOROMSKIJ

We discuss efficient numerical methods for 3D integral transforms representing the Green iterations for the Hartree-Fock/Kohn-Sham equation, based on the tensor-product approximations. We analyse the local convergence of the Newton iteration to solve the Green's function integral formulation of the Hartree-Fock/Kohn-Sham model in electronic structure calculations. We prove the low-separation rank approximations for the arising discrete convolving kernels given by the Coulomb and Yukawa potentials $1/|x|$, and $e^{-\lambda|x|}/|x|$, respectively, with $x \in \mathbb{R}^3$. Complexity analysis of the nonlinear iteration with truncation to the fixed Kronecker tensor-product format is presented. Numerical illustrations demonstrate uniform exponential convergence of tensor approximations in the Tucker and canonical formats.

**High-dimensional PDEs in computational chemistry:
a state-of-the-art review**

CLAUDE LE BRIS

The talk will overview the current challenges in computational chemistry, with an emphasis on high-dimensional problems. Such problems, in the context of chemistry, have definite specificities, as compared to apparently similar problems in other engineering sciences. Some bottlenecks will be indicated, some tracks to overcome the difficulties will be reviewed.

**Computational quantum molecular dynamics using
Hagedorn wavepackets**

CH. LUBICH

(joint work with E. Faou and V. Gradinaru)

We consider the approximation of multi-particle quantum dynamics in the semiclassical regime by Hagedorn wavepackets, which are products of complex Gaussians with polynomials that form an orthonormal L^2 basis and preserve their type under propagation in Schrödinger equations with quadratic potentials. We build a time-reversible, fully explicit time-stepping algorithm to approximate the solution of the Hagedorn wavepacket dynamics. The algorithm is based on a splitting between the kinetic and potential part of the Hamiltonian operator, as well as on a splitting of the potential into its local quadratic approximation and the remainder. The algorithm reduces to the Strang splitting of the Schrödinger equation in the limit of the full basis set, and it is robust in the semiclassical limit. The algorithm allows for the treatment of multi-particle problems by thinning out the basis according to a hyperbolic cross approximation, and of high-dimensional problems by Hartree-type approximations in a moving coordinate frame.

Harmonic and multiscale analysis of and on data in high-dimensions

MAURO MAGGIONI

In many applications one is faced with the task of analyzing large amounts of data, typically embedded in high-dimensional space, but with a lower effective dimensionality, due to physical or statistical constraints. We are interested in studying the geometry of such data sets, modeled as noisy manifolds or graphs, in particular in estimating its intrinsic dimensionality and finding intrinsic coordinate systems on the data. We discuss recent results in these directions, where eigenfunctions of a Laplacian on the data or the associated heat kernel can be used to introduce coordinates with provable guarantees on their bi-Lipschitz distortion. We also discuss ways of studying, fitting, denoising and regularizing functions defined on the data, by using Fourier or a wavelet-like multiscale analysis on the data. We present toy applications to nonlinear image denoising, semisupervised learning on a family of benchmark datasets, and Markov decision processes.

Multivariate Regression and Machine Learning with Sums of Separable Functions

MARTIN J. MOHLENKAMP

(joint work with Gregory Beylkin and Jochen Garcke)

I will present an algorithm for learning a function of many variables from scattered data. The function is approximated by a sum of separable functions, following the paradigm of separated representations. The central fitting algorithm is linear in both the number of data points and the number of variables, and thus is suitable for large data sets in high dimensions.

Tackling Higher Dimensions in Data Mining Using Adaptive Sparse Grids

DIRK PFLÜGER

(joint work with Hans-Joachim Bungartz)

The task of classification in machine learning can be considered as the problem of reconstructing an unknown function represented by randomly sampled function evaluations (training data). Following the regularization network approach, the functional $H[f] = \frac{1}{M} \sum_{i=1}^M (y_i - f(\vec{x}_i))^2 + \lambda \|f\|_K^2$ is to be minimized. Discretizing the feature space yields somehow data independent classification algorithms scaling only linearly in the number of data points, but the curse of dimensionality encountered by conventional discretization schemes has to be overcome. It has been shown [GGT] that this can be achieved for moderate dimensionalities using the sparse grid combination technique. The combination technique has been extended to allow for dimensional adaptivity to be able to cope with higher dimensionalities [Ga].

Our sparse grid approach, however, is to use spatially adaptive sparse grids. It is sometimes considered that this is not feasible in high dimensional settings, but we showed recently [PMB] that this way one can exploit the characteristics of the underlying data, leading to competitive classification results. Grid points are spent mainly in critical regions, thus reducing the number of grid points needed significantly.

Tackling higher dimensional problems (beyond 10) demands further modifications. We show which steps led us to our current classification algorithm, enabling us to successfully classify datasets with 34 and more dimensions. We will focus especially on the handling of the boundary and on the regularization operator $\|f\|_K^2$. Considerations about complexities will be given and empirical results for common benchmark problems will be shown.

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Random Sampling of Sparse Trigonometric Polynomials

HOLGER RAUHUT

The recently discovered principle of compressed sensing says that a high dimensional but sparse vector can be recovered from only a small number of random measurements. We present results of this type in the context of reconstructing sparse trigonometric polynomials from a small number of random samples [Ra07, KR, Ra06, Ra08]. As reconstruction method we use l1-minimization (basis pursuit) as well as the greedy algorithms thresholding and orthogonal matching pursuit. We provide theoretical and numerical evidence of their reconstruction ability.

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Sparse Tensor Product-based Wavelet Compression for Anisotropic Operators

NILS REICH

(joint work with Christoph Schwab)

For a wide class of non-local anisotropic operators we present a new sparse tensor product-based wavelet compression scheme using anisotropic tensor product wavelets. The scheme (asymptotically) reduces the complexity of the corresponding Finite Element stiffness matrix from originally $\mathcal{O}(h^{-2d})$ to essentially optimal $\mathcal{O}(h^{-1}|\log h|^{2(d-1)})$, where h denotes the meshwidth of the Finite Element discretization. It is based on a combination of concepts from [DHS, GOS] and the references therein.

To describe the operators under consideration, we introduce anisotropic symbol classes and present some of their analytical properties (e.g. relaxed smoothness requirements for sparse tensor product convergence).

The work is motivated by Finite Element-based derivatives pricing in multi-dimensional Lévy models as described in [FRS]. The compression techniques are neither limited to Lévy models nor to non-local operators only arising in finance. Numerical results are presented.

Keywords: Wavelet Compression, Sparse Tensor Products, Markov Processes, Lévy Copulas.

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Quadrature of Lipschitz Functionals and Approximation of Distributions

KLAUS RITTER

(joint work with Jakob Creutzig, Steffen Dereich, Thomas Müller-Gronbach)

We study randomized (i.e. Monte Carlo) algorithms to compute expectations of Lipschitz functionals w.r.t. measures on infinite-dimensional spaces, e.g., Gaussian measures or distribution of diffusion processes. We determine the order of minimal errors and corresponding almost optimal algorithms for three different sampling regimes: fixed-subspace-sampling, variable-subspace-sampling, and full-space sampling. It turns out that these minimal errors are closely related to quantization numbers and Kolmogorov widths for the underlying measure. For variable-subspace-sampling suitable multi-level Monte Carlo methods, which have recently been introduced by Giles, turn out to be almost optimal.

On the Sparsity of the Potential Operators in the Electronic Schrödinger Equation

REINHOLD SCHNEIDER

We consider the sparsity of the Hamilton operator corresponding to a quantum mechanical system of N electrons in a electro static field of nucleons. We present estimates for the interaction of Slater determinants built by localized orbitals. Furthermore we consider the interaction of slater determinants built by orthogonal wavelets and proved the s -sparsity of the Galerkin matrix, for $0 < s < S$ $S = \frac{1}{4}$. ($S = \frac{1}{2}$). Using a recently developed adaptive solution strategy, a best n -term convergence rate about $n^{-1/6}$ then could be achieved, according to the recent regularity result of H. Yserentant.. For the kinetic energy, resp. Laplace operator, the sparsity can be imediately deduced by applying the Slater Condon rules. However the Coulomb interaction of two electrons require serious technical efforts.

Convergence Rates for Galerkin FEM for elliptic SPDEs

CH. SCHWAB

We consider the Finite Element Solution of second order elliptic problems in a physical domain $D \subset \mathbb{R}^d$ with spatially inhomogeneous random coefficients.

We present convergence rates and complexity estimates for sparse Galerkin semidiscretization in the probability domain of the random solution. It is parametric in the first M Karhunen-Loève (KL) variables of the input data [TS06].

Two cases are distinguished:

(i) Exponential decay of the input's KL expansion based on [TS]

and

(ii) algebraic decay of the input's KL expansion.

In (i), a “polynomial chaos” type Galerkin discretization is shown to yield spectral convergence rates *in terms of* N_Ω , the number of deterministic elliptic problems to be solved. In (ii), first results on nonlinear approximation rates in terms of N_Ω are available.

Finally, in

(iii) ongoing work [BS] on the total complexity vs. accuracy of (adaptive) tensor Galerkin discretizations in both, stochastic as well as in the deterministic domain D will be addressed.

Sufficient conditions on the joint pdf's of the random field input to ensure better complexity than with (Quasi) Monte Carlo in the probability domain and with Galerkin discretization in D will be identified and implementational issues will be addressed in each case.

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Sparse approximation of high-dimensional degenerate elliptic equations

ENDRE SÜLI

(joint work with Christoph Schwab and Radu-Alexandru Todor)

High-dimensional partial differential equations with nonnegative characteristic form arise in numerous mathematical models in science: mathematical finance and the kinetic theory of polymers are particularly fertile sources of equations of this type.

For such problems, the exponential growth of computational complexity as a function of the dimension d of the problem domain $\Omega \subset \mathbb{R}^d$, the so-called “curse of dimension”, is exacerbated by the fact that the equation may be transport/drift-dominated.

We develop the numerical analysis of stabilized sparse tensor-product finite element methods for such high-dimensional, non-self-adjoint and possibly degenerate second-order partial differential equations.

Our convergence analysis is based on new optimal approximation results in sparse tensor-product spaces, established in [SST]. By tracking the dependence of the various constants on the dimension d and the polynomial degree $p \geq 1$, we also show that there exists a positive integer $L_0 = L_0(p, d)$, which grows linearly with d for each $p \geq 1$, such that, using an $(L + 1)$ -level sparse basis with $L \in [0, L_0]$, the error-constant exhibits *exponential decay* as $d \rightarrow \infty$. In particular, for $L \in [0, L_0]$ the familiar polylogarithmic factor $\log^{d-1}(L + 1)$, featuring in sparse approximation error bounds in the $L^2(\Omega)$ norm, can be absorbed into the exponentially decreasing error constant. This, in turn, shows that for each $p \geq 1$, there is a preasymptotic region $L \in [0, L_0]$, whose size L_0 grows linearly with the dimension d , within which the potentially harmful polylogarithmic factor $\log^{d-1}(L + 1)$ is practically negligible.

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The surprising structure of Gaussian point clouds and its implications for signal processing.

JARED TANNER

(joint work with David L. Donoho)

We will explore connections between the structure of high-dimensional convex polytopes and information acquisition for compressible signals. A classical result in the field of convex polytopes is that if N points are distributed Gaussian i.i.d. at random in dimension $n \ll N$, then only order $\mathcal{O}(\log N)^{n/2}$ of the points are vertices of their convex hull. Recent results show that provided n grows slowly with N , then with high probability all of the points are vertices of its convex hull. More surprisingly, a rich "neighborliness" structure emerges in the faces of the convex hull. One implication of this phenomenon is that an N -vector with k non-zeros can be recovered computationally efficiently from only n random projections with $n = 2ek \log(N/n)$. Alternatively, the best k -term approximation of a signal in any basis can be recovered from $2ek \log(N/n)$ non-adaptive measurements, which is within a log factor of the optimal rate achievable for adaptive sampling. Additional implications for randomized error correcting codes will be presented.

Structured Tensor Decompositions and Approximations

E. TYRTYSHNIKOV

(joint work with I. Oseledets, D. Savostyanov)

The main purpose of the talk is to show how theory and methods recently obtained for rank-structured matrices [GTZ, OOT] develop into theory and methods for rank-structured tensors [T, HKT1, HKT2]. Specifically, we present our 3D cross approximation algorithm together with its application to compression of petabyte-size data and preliminary results of design of iterative methods with rank-structured vectors [OST1, OST2]. We also present some new tensor rank estimates for specifically structured tensors providing a base for structure-preserving approximate iterations (see [HKT2]).

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Adaptive radiative transfer parameterisation schemes utilising spatial and temporal correlations

VICTOR VENEMA

(joint work with Annika Schomburg, Felix Ament, and Clemens Simmer)

Dynamical weather prediction and climate models solve partial differential equations on large 3D grids with millions of grid points. Still these models can only resolve large scale flow and need to model (parameterise) the influence of sub-scale processes that take place on scales below the model resolution. These parameterisations need information from the resolved model fields. We introduce the term 'adaptive parameterisation scheme', for a scheme, which uses spatial and temporal correlations in the resolved geophysical fields to make the parameterisations computationally more efficient. Here the results of two adaptive radiative transfer parameterisations for a numerical weather prediction (NWP) model will be presented as an example. The current radiation scheme is computationally expensive and thus only called once an hour.

In an adaptive parameterisation scheme, the computation is split into a more complex, intrinsic calculation and a simple, adaptive generalisation algorithm. We propose to make an intrinsic calculation at a fraction of the time steps or only in a part of the grid boxes (columns) to reduce the total computational cost. To generalise the results to the full domain, an adaptive generalisation method is used that utilises the results of nearby (in time and space) intrinsic calculations. As intrinsic calculation for our two adaptive schemes, we use a 1 dimensional radiative transfer parameterisation (a so-called δ -two-stream scheme) that computes the radiative fluxes separately for every model column, taking into account the cloud liquid water profile, the cloud cover profile, gases, aerosols and the surface albedo.

The first adaptive scheme comprises an adaptive selection mechanism and a perturbation algorithm. These two components are based on a simple radiation scheme realised by a multivariate linear regression algorithm. For grid points where the regression algorithm predicts large changes, an intrinsic calculation is performed. At the other grid points an increment computed with the regression algorithm is added to the radiation fluxes, in order to account for small changes in the atmospheric column.

The second scheme uses the spatial and temporal correlations in the field. The adaptive generalisation utilises the result of a similar nearby column. The similarity is based on e.g. cloud cover and total column cloud liquid water.

Compared to traditional schemes with a similar CPU-time, the adaptive schemes are more than 40% more accurate. Furthermore, with a only somewhat smaller improvement in the accuracy, the schemes can be a factor three more computationally efficient as traditional schemes [VSAS].

The equations of the schemes and more results can be found in [VSAS].

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Solving, Estimating and Selecting Nonlinear Dynamic Economic Models without the Curse of Dimensionality

VIKTOR WINSCHHEL

(joint work with Markus Krätzig)

We implement in [WK] a framework to estimate the parameters of a general class of high dimensional nonlinear dynamic general equilibrium models. They are formulated in terms of equilibrium conditions $f(s_t, x_t, E_e h(s_t, x_t, e_t, s_{t+1}, x_{t+1}; \theta); \theta) = 0$ with states s , endogeneous policy variables x , random shocks e , parameters θ and rational expectations over the shocks E_e . Together with the state transition equations $s_{t+1} = g(s_t, x_t, e_t; \theta)$ we can solve for the policy functions $x^*(s_t)$. This solution implies the state transition equations $s_{t+1} = g^*(s_t, e_t; \theta)$ and forms together with the measurement equations $y_t = m(s_t, x(s_t), \epsilon_t; \theta)$ a nonlinear state space model, where ϵ are the measurement errors and y the observed variables.

We apply the Smolyak operator for an approximation of the solution based on Chebyshev polynomials. Gaussian quadrature on sparse grids is used for the integrals arising as rational expectations. The Smolyak operator allows for the first time to solve interesting models of substantial size with a global approximation scheme and to overcome the usual local solution methods based on perturbation approaches. The size of calculatable models in our current implementation is around 20 states and policies. Current macroeconomic models with hundreds of states and policies are solved exclusively with perturbation methods.

We evaluate the likelihood of a sample of observables given a parameter vector with a new nonlinear state space filter which is a combination of the new deterministic Smolyak Kalman filter and the standard sequential importance resampling particle filter. This filter substantially decreases the computational burden of the standard particle filter. The posterior density of the parameters is finally estimated by a new Metropolis-Hastings algorithm with parallel sequences and a feature from a global genetic maximizer. The parallel sequences improve the algorithm's global maximization properties, simplify the choice of the innovation variances, allow for unbiased convergence diagnostic tests and for a simple implementation on parallel computers. We also calculate the marginal likelihood from the Metropolis-Hastings sequences as a model selection criterion. Finally, we provide the open source software JBendge (Java based Bayesian Estimation of Nonlinear Dynamic General Equilibrium models) for the solution, estimation and selection of the general class of economic models. JBendge has a modern graphical user interface, an object oriented architecture and runs on parallel computers.

The first application of sparse grids in the theoretical macroeconomic literature was [KK]. A first application in microeconometrics is [HW].

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